

HIGHER TORSION AND SECONDARY TRANSFER OF UNIPOTENT BUNDLES

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ABSTRACT. Given a unipotent bundle of smooth manifolds we construct its secondary transfer map and show that this map determines the higher smooth torsion of the bundle. This approach to higher torsion provides a new perspective on some of its properties. In particular it yields in a natural way a formula for torsion of a composition of two bundles.

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1. INTRODUCTION

By a smooth bundle of manifolds we will understand here a smooth submersion $p: E \rightarrow B$ where E and B are smooth compact manifolds. A smooth bundle with the fiber F is unipotent if B is path connected and the graded vector space $H_*(F; \mathbb{Q})$ admits a filtration such that $\pi_1 B$ acts trivially on the filtration quotients. Igusa and Klein [8, 11] showed using fiberwise Morse theory that to any unipotent bundle one can associate the higher torsion invariant which depends not only on the topological structure of p , but also on its smooth structure. Higher torsion proved to be a useful tool in the study of smooth bundles. In [7] Igusa showed, for example, that it can be used to detect exotic disc bundles constructed by Hatcher.

In [2] and [1] the present authors in collaboration with Klein and Williams extended ideas of Dwyer, Weiss, and Williams [5] to obtain an alternative construction of torsion of unipotent bundles based on the machinery of homotopy theory. This construction can be briefly described as follows. Let $K(\mathbb{Q})$ be the infinite loop space underlying the algebraic K -theory spectrum of the field of rational numbers. Given a smooth bundle $p: E \rightarrow B$ we can construct a map $c_p: B \rightarrow K(\mathbb{Q})$ which, roughly speaking, assigns to each $b \in B$ the point of $K(\mathbb{Q})$ represented by the singular chain complex $C_*(p^{-1}(b); \mathbb{Q})$. The smooth Riemann-Roch theorem of [5] implies that c_p admits a factorization

$$\begin{array}{ccc} & & Q(E_+) \\ & \nearrow p^! & \downarrow \lambda_E \\ B & \xrightarrow{c_p} & K(\mathbb{Q}) \end{array}$$

where $Q(E_+) = \Omega^\infty \Sigma^\infty(E_+)$, $p^!$ is the Becker-Gottlieb transfer and λ_E is the linearization map (3.1).

If p is a unipotent bundle then the map c_p is homotopic via a preferred homotopy to a constant map. As a consequence we obtain a lift of $p^!$ to the space $\text{Wh}_s^{\mathbb{Q}}(E)$ which is the homotopy fiber of λ_E . This lift $\tau^s(p)$ is the smooth torsion of the bundle p .

The homopy class of $\tau^s(p)$ is an invariant of the smooth structure of p in the following sense. If $p': E' \rightarrow B$ is another smooth bundle and $f: E' \rightarrow E$ is a smooth bundle map then f induces a map

$$f_*: \text{Wh}_s^{\mathbb{Q}}(E') \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$$

The map $f_* \tau^s(p')$ need not be homotopic to $\tau^s(p)$ in general, but this property does hold provided that f is a fiberwise diffeomorphism of bundles.

The map $\tau^s(p)$ gives rise to a certain cohomology class

$$t^s(p) \in \bigoplus_{k \geq 0} H^{4k}(B; \mathbb{R})$$

[1, 4.10] which we will call the cohomological torsion of the bundle p .

In [9, Section 9] Igusa showed that the cohomological torsion of the composition pq can be, in some cases, computed from the torsion of the bundles p and q . Namely, if q is an oriented linear sphere bundle then we have

$$(1-1) \quad t^s(pq) = \chi(F_q)t^s(p) + \text{tr}_B^E(t^s(q))$$

where $\chi(F_q) \in \mathbb{Z}$ is the Euler characteristic of the fiber of q and

$$\text{tr}_B^E: H^*(E; \mathbb{R}) \rightarrow H^*(B; \mathbb{R})$$

is the transfer map associated to p ¹. In [9] Igusa calls the formula (1-1) the transfer axiom and shows that taken together with a few other properties it uniquely determines the cohomological torsion.

Igusa's arguments can be used to show that the formula (1-1) holds under more general conditions on p and q , e.g. if dimensions of fibers of these bundles have the same parity. In [1, Thm 7.1] we verified that the same is true in the case when p is an arbitrary unipotent bundle and q satisfies the assumptions of the Leray-Hirsch isomorphism theorem. One of our goals in this paper is to show that this formula holds in general:

1.1. Theorem. *The formula (1-1) holds for any unipotent bundles $p: E \rightarrow B$ and $q: D \rightarrow E$.*

In order to prove Theorem 1.1 we develop a new construction of smooth torsion based on the notion of the secondary transfer of unipotent bundles. The starting point for this construction is the following

1.2. Theorem. *Given a smooth bundle of compact manifolds $p: E \rightarrow B$ with fiber F_p consider the diagram*

$$(1-2) \quad \begin{array}{ccc} Q(B_+) & \xrightarrow{Q(p^!)} & Q(E_+) \\ \lambda_B \downarrow & & \downarrow \lambda_E \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) \end{array}$$

where the lower horizontal map is given by the multiplication by the Euler characteristic $\chi(F_p) \in \mathbb{Z}$ of F_p and the upper horizontal map is the Becker-Gottlieb

¹In [14] Ma showed that an analogous formula is satisfied by the higher analytical torsion of Bismitt and Lott.

transfer of p . If p is a unipotent bundle then this diagram commutes up to a preferred homotopy

$$\eta_p: \mathcal{Q}(B_+) \times [0, 1] \rightarrow K(\mathbb{Q})$$

For a unipotent bundle $p: E \rightarrow B$ the homotopy η_p defines a map of homotopy fibers

$$\mathrm{Wh}_s^{\mathbb{Q}}(p^!): \mathrm{Wh}_s^{\mathbb{Q}}(B) \rightarrow \mathrm{Wh}_s^{\mathbb{Q}}(E)$$

This map is the smooth secondary transfer of the bundle p .

The smooth secondary transfer shares some of the basic properties of the Becker-Gottlieb transfer. It is additive (7.3) and it preserves composition of bundles:

1.3. Theorem. *If $p: E \rightarrow B$ and $q: D \rightarrow E$ are unipotent bundles then*

$$\mathrm{Wh}_s^{\mathbb{Q}}((pq)^!) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(q^!) \circ \mathrm{Wh}_s^{\mathbb{Q}}(p^!)$$

The relationship between the smooth secondary transfer and the smooth torsion is as follows. If B is a compact, smooth manifold then the identity map $\mathrm{id}_B: B \rightarrow B$ is a unipotent bundle. We have

1.4. Theorem. *If $p: E \rightarrow B$ is a unipotent bundle then*

$$\tau^s(p) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(p^!) \circ \tau^s(\mathrm{id}_B)$$

This shows that the smooth secondary transfer of a unipotent bundle determines the smooth torsion of the bundle.

Combining Theorems 1.4 and 1.3 we obtain

1.5. Corollary. *If $p: E \rightarrow B$ and $q: D \rightarrow E$ are unipotent bundles then*

$$\tau^s(pq) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(q^!) \circ \tau^s(p)$$

Theorem 1.1 can be obtained as a direct consequence of this fact. Notice that in this way we exhibit the simple principle underlying the formula (1-1): the torsion of a composition of bundles p and q is a composition of two maps, one depending on p and the other on q .

1.6. Organization of the paper. Section 2 contains a brief review of Waldhausen categories which provide the technical setting for most of the construction of this paper. In Section 3 we take a closer look at the statement of Theorem 1.2. We show that this theorem follows from a more general fact (Proposition 3.5) which applies to all unipotent fibrations, and not just unipotent bundles. The proof of Proposition 3.5 depends on a theorem of Brown,² which states that the singular chain complex of the total space of a fibration is quasi-isomorphic to a twisted tensor product of

² See also [10] for a nice description of the relationship of Brown's work to the higher torsion of Igusa-Klein.

the chain complexes of the base and the fiber. In Section 4 we give an overview of this result and describe some properties of Brown's quasi-isomorphism. Section 5 contains the proof of Proposition 3.5, which lets us complete the construction the smooth secondary transfer $\text{Wh}_s^{\mathbb{Q}}(p')$ of a unipotent bundle p . We also construct there the homotopy secondary transfer $\text{Wh}_h^{\mathbb{Q}}(p')$ that exists for any unipotent fibration p . The main difference between these two notions of secondary transfers is that while the smooth transfer depends on the smooth structure of the bundle, the homotopy transfer is in fact a fiberwise homotopy invariant (§6). In Section 7 we obtain additivity formulas for both smooth and homotopy secondary transfers, and in Sections 8 and 9 we study secondary transfers of compositions of unipotent bundles and fibrations. This leads us to the proof of Theorem 1.3 and of its analog (8.1) for the homotopy secondary transfer. Finally, in Section 10 we prove the relationship between the smooth secondary transfer and the smooth torsion of a bundle p described by Theorem 1.4. We also show that it implies the statement on Theorem 1.1.

Several argument of the paper involve constructions of maps between homotopy fibers and constructions of homotopies of such maps. The appendix (§11) gives the basic outline of such constructions.

2. TECHNICAL SETUP

A great majority of constructions described in this paper is set within the realm of Waldhausen categories [16], i.e. categories with equivalences and cofibrations satisfying certain axioms. Our basic setup in this respect will be largely the same as that of [1, Section 3], so we summarize it here only briefly. Given a Waldhausen category \mathcal{C} we will denote by $K(\mathcal{C})$ the K -theory of \mathcal{C} . The standard construction of $K(\mathcal{C})$ proceeds using the Waldhausen S_{\bullet} -construction. For our purposes it will be more convenient though to use its variant, the S'_{\bullet} -construction described by Blumberg and Mandel in [3, §2].

A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of Waldhausen categories is exact if it preserves weak equivalences, cofibrations and pushouts of diagrams

$$(2-1) \quad c' \leftarrow c \rightarrow c''$$

where one of the morphisms is a cofibration. Any such functor induces a map of infinite loop spaces $K(F): K(\mathcal{C}) \rightarrow K(\mathcal{D})$. The advantage of working with the S'_{\bullet} -construction is that we can obtain the map $K(F)$ under more relaxed assumptions on the functor F . Namely, following the terminology of [1, 3.4] we will also say that a functor F is almost exact if it preserves weak equivalences and cofibrations, and if it preserves pushouts of diagrams (2-1) up to a weak equivalence. An almost exact functor induces a functor of simplicial categories $F: S'_{\bullet}\mathcal{C} \rightarrow S'_{\bullet}\mathcal{D}$, and so it yields a map $K(F): K(\mathcal{C}) \rightarrow K(\mathcal{D})$.

We will work mainly with two specific instances of Waldhausen categories. For a topological space X the category $\mathcal{R}^{fd}(X)$ has as its objects homotopy finitely dominated retractive spaces over X , while its morphisms are maps of retractive spaces. It is a Waldhausen category with cofibrations given by closed embeddings having the homotopy extension property and weak equivalences defined as homotopy equivalences. The K -theory of $\mathcal{R}^{fd}(X)$ is the Waldhausen algebraic K -theory space of X denoted it by $A(X)$ ³.

Next, let $\mathcal{C}h^{fd}(\mathbb{Q})$ denote the category of homotopy finitely dominated chain complexes of \mathbb{Q} -vector spaces. This is a Waldhausen category with degreewise monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. We will denote by $K(\mathbb{Q})$ the K -theory of $\mathcal{C}h^{fd}(\mathbb{Q})$. This space describes the algebraic K -theory of \mathbb{Q} : $K(\mathbb{Q}) \simeq \Omega BGL(\mathbb{Q})^+$.

As we have already mentioned exact and almost exact functors between Waldhausen categories define maps between their associated K -theories. We will frequently need to construct homotopies of such maps. There are two main sources of such constructions. First, if $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are (almost) exact functors then a natural weak equivalence $\varphi: F \Rightarrow G$ defines a homotopy $K(\varphi)$ between the induced maps $K(F)$ and $K(G)$. Second, if $F_i: \mathcal{C} \rightarrow \mathcal{D}$ are (almost) exact functors for $i = 0, 1, 2$ and $\varphi: F_0 \Rightarrow F_1, \psi: F_1 \Rightarrow F_2$ are natural transformations such that

$$F_0(c) \xrightarrow{\varphi} F_1(c) \xrightarrow{\psi} F_2(c)$$

is a cofibration sequence for each $c \in \mathcal{C}$, then the additivity theorem of Waldhausen [16, Theorem 1.4.2] provides a homotopy between the map $K(F_1)$ and $K(F_0 \vee F_2)$. A convenient combinatorial construction of this homotopy has been given by Grayson in [6].

Our work will require us to go a step further beyond homotopies and consider homotopies of homotopies. If $f_1, f_2: X \rightarrow Y$ are two maps between topological spaces and $h_1, h_2: X \times I \rightarrow Y$ are homotopies between f_1 and f_2 then by a homotopy of homotopies we will understand a map $H: X \times I \times I \rightarrow Y$ such that

$$\begin{cases} H|_{X \times \{i\} \times \{t\}} &= f_i \quad \text{for } i = 0, 1, \text{ and } t \in I \\ H|_{X \times I \times \{j\}} &= h_j \quad \text{for } j = 0, 1 \end{cases}$$

In our constructions of homotopies of homotopies we will always have $Y = K(\mathbb{Q})$. The homotopies of homotopies we will consider will come from the following sources:

³If X is a path connected space then by abuse of notation by $\mathcal{R}^{fd}(X)$ we will understand the Waldhausen category of *path connected* retractive spaces over X . From the perspective of K -theory this change is of little consequence: the functor that embeds this category into the category of all retractive spaces over X induces a homotopy equivalence of the associated K -theory spaces.

- If $F_1, F_2: \mathcal{C} \rightarrow \mathcal{R}^{fd}(\mathbb{Q})$ are (almost) exact functors, $\varphi_1, \varphi_2: F_1 \Rightarrow F_2$ are natural weak equivalences and Φ is a natural chain homotopy between φ_1 and φ_2 , then Φ defines a homotopy of homotopies $K(\Phi)$ between $K(\varphi_1)$ and $K(\varphi_2)$.
- Assume that we have functors $F_i, G_i: \mathcal{C} \rightarrow \mathcal{R}^{fd}(\mathbb{Q})$ ($i = 0, 1, 2$) and that we have a commutative diagram of natural transformations

$$\begin{array}{ccccc} F_0 & \xRightarrow{\varphi} & F_1 & \xRightarrow{\psi} & F_2 \\ \eta_0 \downarrow & & \downarrow \eta_1 & & \downarrow \eta_2 \\ G_0 & \xRightarrow{\varphi'} & G_1 & \xRightarrow{\psi'} & G_2 \end{array}$$

where both rows are cofibration sequences. This yields a diagram

$$\begin{array}{ccc} K(F_1) & \xrightarrow{\mathfrak{U}} & K(F_0) + K(F_2) \\ K(\eta_1) \downarrow & & \downarrow K(\eta_0) + K(\eta_2) \\ K(G_1) & \xrightarrow{\mathfrak{U}} & K(G_0) + K(G_2) \end{array}$$

In this diagram every vertex represents a map $K(\mathcal{C}) \rightarrow K(\mathbb{Q})$ and each edge is a homotopy of such maps. In this setting there exists a homotopy of homotopies that fills this diagram, i.e. a homotopy of homotopies between the concatenation of \mathfrak{U} with $K(\eta_0) + K(\eta_2)$ and the concatenation of $K(\eta_1)$ with \mathfrak{U} .

3. THE LINEARIZATION MAP AND THE \mathbf{A} -THEORY TRANSFER

3.1. In preparation for the proof of Theorem 1.2 we start this section by reviewing briefly the construction of the linearization map $\lambda_B: Q(B_+) \rightarrow K(\mathbb{Q})$. For an arbitrary space B we have an assembly map $a_B: Q(B_+) \rightarrow A(B)$. If B is a smooth manifold this map has a convenient combinatorial description that relies on Waldhausen's construction of $Q(B_+)$ using “partitions” [2, §3].

Next, for any space B we have the A -theory linearization map $\lambda_B^h: A(B) \rightarrow K(\mathbb{Q})$. The construction of this map proceeds as follows. Recall that the spaces $A(B)$ and $K(\mathbb{Q})$ are constructed from the Waldhausen categories $\mathcal{R}^{fd}(B)$ and $\mathcal{C}h^{fd}(\mathbb{Q})$. Consider the functor

$$\Lambda_B: \mathcal{R}^{fd}(B) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$$

that assigns to a retractive space X the relative singular chain complex $C_*(X, B)$. This functor is almost exact, so it induces a map $\lambda_B^h: A(B) \rightarrow K(\mathbb{Q})$. The linearization map $\lambda_B: Q(B_+) \rightarrow K(\mathbb{Q})$ is given by the composition

$$\lambda_B = \lambda_B^h a_B$$

3.2. Assume now that we have a fibration $p: E \rightarrow B$. For a retractive space $X \in \mathcal{R}^{fd}(B)$ let p^*X denote the pullback

$$p^*X := \lim(X \rightarrow B \xleftarrow{p} E)$$

The assignment $X \mapsto p^*X$ defines an exact functor $\mathcal{R}^{fd}(B) \rightarrow \mathcal{R}^{fd}(E)$. We will call the induced map $A(p^!): A(B) \rightarrow A(E)$ the A -theory transfer of p .

3.3. Let $p: E \rightarrow B$ be a fibration with a homotopy finitely dominated fiber F_p . The maps described above can be assembled into a diagram

$$(3-1) \quad \begin{array}{ccc} Q(B_+) & \xrightarrow{Q(p^!)} & Q(E_+) \\ a_B \downarrow & & \downarrow a_E \\ A(B) & \xrightarrow{A(p^!)} & A(E) \\ \lambda_B^h \downarrow & & \downarrow \lambda_E^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) \end{array}$$

The outer rectangle in this diagram coincides with the diagram (1-2). If $p: E \rightarrow B$ is a smooth bundle then using the combinatorial construction of the Becker-Gottlieb transfer described in [2, §4.2] we get that the upper square in the diagram (3-1) commutes.⁴ In order to obtain Theorem 1.2 it is then enough to show that the lower square of (3-1) is homotopy commutative. We will show that this fact holds for any unipotent fibration p :

3.4. Definition. A fibration $p: E \rightarrow B$ is *unipotent* if B is a path connected space, both B and the fiber F_p of p have the homotopy type of a finite CW-complex, and $H_*(F_p)$ admits a filtration by $\pi_1 B$ -modules such that the action of $\pi_1 B$ on the filtration quotients is trivial.

3.5. Proposition. *Let $p: E \rightarrow B$ be a unipotent fibration with fiber F_p . The diagram*

$$(3-2) \quad \begin{array}{ccc} A(B) & \xrightarrow{A(p^!)} & A(E) \\ \lambda_B^h \downarrow & & \downarrow \lambda_E^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) \end{array}$$

commutes up to a preferred homotopy $\eta_p^h: A(B) \times I \rightarrow K(\mathbb{Q})$.

The proof of Proposition 3.5 will be given in Section 5.

⁴While all maps in the upper square of (3-1) are defined for any fibration, smoothness of p is essential for its commutativity. This diagram will not commute, in general, when p is a fibration. See e.g. the proof of Theorem F in [12]

4. TWISTED TENSOR PRODUCTS

The main ingredient of the proof of Proposition 3.5 is the fact that given a unipotent fibration $p: E \rightarrow B$ with fiber F_p we can describe a preferred path in $K(\mathbb{Q})$ joining the points represented by the chain complex $C_*(E)$ and the chain complex $C_*(B) \otimes H_*(F_p)$. This path is natural enough that it gives rise to a homotopy filling the diagram (3-2). Our main tool in the construction of this path will be the theorem of Brown [4] which shows that the chain complex $C_*(E)$ is quasi-isomorphic to a twisted tensor product $C_*(B) \otimes_{\varphi_p} H_*(F_p)$. We begin this section by reviewing the relevant notions in homological algebra. Subsequently we describe Brown's result and develop some of its properties that we will need later on.

4.1. Twisting cochains and twisted tensor products. Let A be a differential graded \mathbb{Q} -algebra with multiplication $\mu: A \otimes A \rightarrow A$, and let K be a d.g. \mathbb{Q} -coalgebra with comultiplication $\nabla: K \rightarrow K \otimes K$. Given homomorphisms of graded vector spaces $\varphi, \psi: K \rightarrow A$ the cup product $\varphi \cup \psi: K \rightarrow A$ is given by the formula

$$\varphi \cup \psi := \mu(\varphi \otimes \psi) \nabla$$

If M is a left A -module with multiplication $\nu: A \otimes M \rightarrow M$ then for φ as above and $c \in K \otimes M$ the cap product $\varphi \cap c \in K \otimes M$ is given by

$$\varphi \cap c := (\text{id}_K \otimes \nu)(\text{id}_K \otimes \varphi \otimes \text{id}_M)(\nabla \otimes \text{id}_M)(c)$$

The map

$$\varphi \cap -: K \otimes M \rightarrow K \otimes M, \quad c \mapsto \varphi \cap c$$

is a homomorphism of graded vector spaces.

4.2. Definition. Let A, K, M be respectively a d.g. \mathbb{Q} -algebra, coalgebra, and a left A -module as above.

(i) A twisting cochain is a homomorphism of graded vector spaces $\varphi: K \rightarrow A$ that lowers grading by 1 and satisfies the identity

$$\partial\varphi - \varphi\partial + \varphi \cup \varphi = 0$$

(ii) If $\varphi: K \rightarrow A$ is a twisting cochain then the twisted tensor product $K \otimes_{\varphi} M$ is a chain complex such that $K \otimes_{\varphi} M = K \otimes M$ as a graded vector space, and the differential in $K \otimes_{\varphi} M$ is given by

$$(4-1) \quad \partial_{\varphi} := \partial \otimes \text{id} + \text{id} \otimes \partial + \varphi \cap -$$

4.3. Twisted chain complex of a fibration. Let X be a topological space with the basepoint x_0 . By $C'_*(X)$ we will denote the subcomplex of the singular chain complex $C_*(X)$ generated by all singular simplices $\sigma: \Delta^n \rightarrow X$ that send all vertices of

Δ^n into x_0 . If X is a path connected space then the inclusion $C'_*(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence. The chain complex $C'_*(X)$ can be equipped with the usual d.g. coalgebra structure with comultiplication $\nabla: C'_*(X) \otimes C'_*(X) \rightarrow C'_*(X)$ defined by $\nabla(\sigma) = \sum_{i=0}^n f_i(\sigma) \otimes b_{n-i}(\sigma)$ where $\sigma \in C'_n(X)$ is a singular n -simplex and $f_i(\sigma)$, $b_{n-i}(\sigma)$ denote, respectively, the front i -th face and the back $(n-i)$ -th face of σ .

For a space X we can also consider its associated d.g. homology algebra $\text{End}(H_*(X))$ defined as follows. Let $\text{End}_n(H_*(X))$ denote the vector space of all maps of graded vector spaces $H_*(X) \rightarrow H_*(X)$ that increase the grading by n , and let

$$\text{End}(H_*(X)) = \bigoplus_{n \geq 0} \text{End}_n(H_*(X))$$

We view $\text{End}(H_*(X))$ as a chain complex with trivial differentials. The d.g. algebra structure on $\text{End}(H_*(X))$ comes from composition of maps. Naturally $H_*(X)$ is a module over this d.g. algebra.

The main result of [4] is that given a fibration $p: E \rightarrow B$ with a path connected base space and a fiber F_p we can find a twisting cochain $\varphi_p: C'_*(B) \rightarrow \text{End}(H_*(F_p))$ such that the twisted tensor product $C'_*(B) \otimes_{\varphi_p} H_*(F_p)$ is naturally quasi-isomorphic to $C_*(E)$. For our purposes it will be convenient to state this fact in the following form. Let \mathcal{S}_* denote the category of path connected, pointed spaces, and $\mathcal{S}_* \downarrow B$ be the over category of \mathcal{S}_* over a space B . Given a fibration $p: E \rightarrow B$ and an object $X \in \mathcal{S}_* \downarrow B$ denote by $p_X: p^*X \rightarrow X$ the fibration induced from p .

4.4. Theorem. *Let $p: E \rightarrow B$ be a fibration with a path connected base B and a fiber F_p .*

1) *For every $X \in \mathcal{S}_* \downarrow B$ there exists a twisting cochain*

$$\varphi_{p_X}: C'_*(X) \rightarrow \text{End}(H_*(F_p))$$

and a quasi-isomorphism

$$\beta_{p_X}: C_*(p^*X) \xrightarrow{\sim} C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$$

2) *On $C'_1(X)$ the twisting cochain $\varphi_{p_X}: C'_1(X) \rightarrow \text{End}_0(H_*(F_p))$ is given as follows. If σ is a singular simplex in $C'_1(X)$ then σ is a loop in X , and so it represents an element $[\sigma] \in \pi_1 X$. For $z \in H_*(F_p)$ we have*

$$\varphi_{p_X}(\sigma)(z) = [\sigma]z - z$$

where the product $[\sigma]z$ is defined by the action of $\pi_1 X$ on $H_(F_p)$.*

3) *The assignment $X \mapsto C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$ defines a functor*

$$F: \mathcal{S}_* \downarrow B \rightarrow \text{Ch}(\mathbb{Q})$$

where $\text{Ch}(\mathbb{Q})$ is the category of chain complexes over \mathbb{Q} . If $f: X \rightarrow Y$ is a morphism in $\mathcal{S}_ \downarrow B$ then $F(f) = f_* \otimes \text{id}_{H_*(F_p)}$.*

4) The quasi-isomorphisms β_{p_X} define a natural transformation of functors.

4.5. Notation. For simplicity from now on we will write $C'_*(X) \otimes_{p_X} H_*(F_p)$ to denote the complex $C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$.

4.6. While we refer to Brown's paper [4] for the proof of Theorem 4.4, a few comments will be useful later on. Brown constructs the quasi-isomorphisms β_{p_X} in two stages. First, he shows that given a path connected space X one can construct in a natural way a twisting cochain $\psi_X: C'_*(X) \rightarrow C_*(\Omega X)$ [4, Theorem 4.1]. The action of ΩX on the fiber F_p of p_X defines a $C_*(\Omega X)$ -module structure on $C_*(F_p)$. Brown shows [4, Theorem 4.2] that the twisted tensor product $C'_*(X) \otimes_{\psi_X} C_*(F_p)$ is chain homotopic to $C_*(p^*X)$ via a chain homotopy that is natural in X . A minor technical point here is that in order to get a suitable action of ΩX on F_p one needs to specify a weakly transitive lifting function for the fibration p . This can be taken care of by first replacing the fibration $p: E \rightarrow B$ by the homotopy equivalent fibration $\tilde{p}: PB \times_B E \rightarrow B$ where PB is the space of Moore paths in B . The fibration \tilde{p} admits a canonical lifting function [4, p.225] which can be used to get a chain homotopy equivalence

$$C_*(p^*X) \xrightarrow{\sim} C_*(\tilde{p}^*X) \xrightarrow{\sim} C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}})$$

where $F_{\tilde{p}}$ is the fiber of \tilde{p} . Since $F_{\tilde{p}} \simeq F_p$ we have $C_*(F_{\tilde{p}}) \simeq C_*(F_p)$.⁵

To complete the construction of β_X it suffices to show that we have quasi-isomorphisms

$$C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) \xrightarrow{\sim} C'_*(X) \otimes_{p_X} H_*(F_p)$$

One can proceed as follows. Using the fact that we deal here with chain complexes over a field we can find chain maps

$$j_{F_p}: H_*(F_p) \rightleftarrows C_*(F_{\tilde{p}}): r_{F_p}$$

such that for $z \in H_*(F_p) \cong H_*(F_{\tilde{p}})$ the element $j_{F_p}(z) \in C_*(F_{\tilde{p}})$ is a chain representing z , $r_{F_p} j_{F_p} = \text{id}_{H_*(F_p)}$, and $j_{F_p} r_{F_p} \simeq \text{id}_{C_*(F_{\tilde{p}})}$. The maps j_{F_p} and r_{F_p} define a strong deformation retraction of (untwisted) tensor products

$$\text{id} \otimes j_{F_p}: C'_*(X) \otimes H_*(F_p) \rightleftarrows C'_*(X) \otimes C_*(F_{\tilde{p}}): \text{id} \otimes r_{F_p}$$

The Basic Perturbation Lemma (see e.g. [13, 2.6]) shows that in such situation there is a twisting cochain $\varphi_{p_X}: C'_*(X) \rightarrow \text{End}(H_*(F_p))$ and a strong deformation retraction of twisted tensor products

$$(\text{id} \otimes j_{F_p})^\infty: C'_*(X) \otimes_{p_X} H_*(F_p) \rightleftarrows C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}): (\text{id} \otimes r_{F_p})^\infty$$

Following our convention (4.5) by $C'_*(X) \otimes_{p_X} H_*(F_p)$ we denote here the complex $C'_*(X) \otimes_{\varphi_{p_X}} H_*(F_p)$. We define β_{p_X} as the composition

$$\beta_{p_X}: C_*(p^*X) \longrightarrow C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) \xrightarrow{(\text{id} \otimes r_{F_p})^\infty} C'_*(X) \otimes_{p_X} H_*(F_p)$$

⁵In [4] Brown gives a quasi-isomorphism going in the opposite direction, $C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) \xrightarrow{\sim} C_*(\tilde{p}^*X)$. However, since his argument relies on the method of acyclic models it also produces a natural homotopy inverse of that map, and we work here with this inverse for convenience.

4.7. Homological filtration. Let $\varphi: K \rightarrow A$ be a twisting cochain and let M be an A -module. Directly from the definition of a twisted tensor product it follows that the chain complex $K \otimes_\varphi M$ admits an increasing filtration

$$U_0 \subset U_1 \subset \cdots \subset K \otimes_\varphi M$$

where $U_n := (\bigoplus_{q \leq n} K_q) \otimes_\varphi M$. In the case where M has trivial differentials we have also a decreasing filtration

$$K \otimes_\varphi M = L_0 \supset L_1 \supset \cdots$$

given by $L_n := K \otimes_\varphi (\bigoplus_{q \geq n} M_q)$. Since we will consider this filtration in the situation where M is the homology of some chain complex we will call it the homological filtration of $K \otimes_\varphi M$.

Let $p: E \rightarrow B$ be a fibration with fiber F_p , let $X \in \mathcal{S}_* \downarrow B$, and let $\{L_n(p_X)\}$ denote the homological filtration of the chain complex $C'_*(X) \otimes_{p_X} H_*(F_p)$. We will need an explicit description of the quotients $L_n(p_X)/L_{n+1}(p_X)$. On the level of graded vector spaces we have isomorphisms

$$L_n(p_X)/L_{n+1}(p_X) \cong C'_*(X) \otimes H_n(F_p)$$

In order to describe the differential in the filtration quotients notice that the differential in $C'_*(X) \otimes_{p_X} H_*(F_p)$ is given by

$$\partial(\sigma \otimes z) = \partial\sigma \otimes z + \sum_{i=0}^n f_i(\sigma) \otimes \varphi_{p_X}(b_{n-i}(\sigma))(z)$$

where σ is a singular simplex in $C'_*(X)$, $z \in H_*(F_p)$ and $f_i(\sigma)$, $b_{n-i}(\sigma)$ are the i -th front face and the $(n-i)$ -th back face of σ . Using part 2) of Theorem 4.4 we get from here

$$\partial(\sigma \otimes z) = \partial\sigma \otimes z + f_{n-1}(\sigma) \otimes ([b_1(\sigma)]z - z) \pmod{L_{n+1}(p_X)}$$

As a consequence we obtain

4.8. Proposition. *Let $p: E \rightarrow B$ be a fibration with a fiber F_p . For $X \in \mathcal{S}_* \downarrow B$ let $C'_*(X) \otimes_{\pi_1 X} H_n(F_p)$ denote the chain complex such that*

$$(C'_*(X) \otimes_{\pi_1 X} H_n(F_p))_k = C'_{k-n}(X) \otimes H_n(F_p)$$

and with differential given by

$$\partial(\sigma \otimes z) = \partial\sigma \otimes z + f_{n-1}(\sigma) \otimes ([b_1(\sigma)]z - z)$$

for a singular simplex $\sigma \in C'_(X)$ and $z \in H_n(F_p)$. We have a canonical isomorphism*

$$C'_*(X) \otimes_{\pi_1 X} H_n(F_p) \cong L_n(p_X)/L_{n+1}(p_X)$$

4.9. Maps of fibrations. Assume that that we have a map of fibrations over B :

$$\begin{array}{ccc} E & \xrightarrow{g} & D \\ & \searrow p & \swarrow q \\ & B & \end{array}$$

For $X \in \mathcal{S}_* \downarrow B$ let $g_X: p^*X \rightarrow q^*X$ be the map of the induced fibrations over X . Consider the diagram

$$\begin{array}{ccc} C_*(p^*X) & \xrightarrow{g_X} & C_*(q^*X) \\ \beta_{p_X} \downarrow & & \downarrow \beta_{q_X} \\ C'_*(X) \otimes_{p_X} H_*(F_p) & \dashrightarrow & C'_*(X) \otimes_{q_X} H_*(F_q) \end{array}$$

where F_p, F_q denote, respectively, the fibers of p and q . We would like to construct a natural lower horizontal map that makes this diagram commute up to a homotopy. The obvious candidate for such map is $\text{id} \otimes (g|_{F_p})_*$, where the homomorphism $(g|_{F_p})_*: H_*(F_p) \rightarrow H_*(F_q)$ is induced by restriction of g to the fibers, but this map is not a chain map in general. We can, however, proceed as follows. By the construction of quasi-isomorphisms β_{p_X} and β_{q_X} (4.6) we have a diagram

$$(4-2) \quad \begin{array}{ccc} C_*(p^*X) & \xrightarrow{g_X} & C_*(q^*X) \\ \simeq \downarrow & & \downarrow \simeq \\ C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}) & \xrightarrow{\text{id} \otimes (g|_{F_{\tilde{p}}})_*} & C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{q}}) \\ (\text{id} \otimes r_{F_p})^\infty \downarrow & & \downarrow (\text{id} \otimes r_{F_q})^\infty \\ C'_*(X) \otimes_{p_X} H_*(F_p) & \dashrightarrow^{g_X^\infty} & C'_*(X) \otimes_{q_X} H_*(F_q) \end{array}$$

The compositions of the vertical maps give β_{p_X} and β_{q_X} . The upper square commutes by the naturality properties of Brown's theorem [4, Theorem 4.2]. Recall that the map $(\text{id} \otimes r_{F_p})^\infty$ is a part of the strong deformation retraction data

$$(\text{id} \otimes j_{F_p})^\infty: C'_*(X) \otimes_{p_X} H_*(F_p) \rightleftarrows C'_*(X) \otimes_{\psi_X} C_*(F_{\tilde{p}}): (\text{id} \otimes r_{F_p})^\infty$$

Define a map $g_X^\infty: C'_*(X) \otimes_{p_X} H_*(F_p) \rightarrow C'_*(X) \otimes_{q_X} H_*(F_q)$ by

$$g_X^\infty := (\text{id} \otimes r_{F_q})^\infty \circ (\text{id} \otimes g_*) \circ (\text{id} \otimes j_{F_p})^\infty$$

The lower square in the diagram (4-2) commutes then up to a chain homotopy. The Basic Perturbation Lemma gives explicit formulas for this chain homotopy and for the maps $(\text{id} \otimes r_{F_q})^\infty$ and $(\text{id} \otimes j_{F_p})^\infty$. Direct computations involving these formulas yield the following

4.10. Proposition. *Let B be a pointed, path connected space. Let $p: E \rightarrow B$ and $q: D \rightarrow B$ be fibrations, and let $g: E \rightarrow D$ be map of fibrations.*

1) *The maps g_X^∞ define a natural transformation of functors $\mathcal{S}_* \downarrow B \rightarrow \mathcal{Ch}(\mathbb{Q})$.*

2) *The diagram*

$$\begin{array}{ccc} C_*(p^*X) & \xrightarrow{g_X^*} & C_*(q^*X) \\ \beta_{p_X} \downarrow \simeq & & \simeq \downarrow \beta_{q_X} \\ C'_*(X) \otimes_{p_X} H_*(F_p) & \xrightarrow{g_X^\infty} & C'_*(X) \otimes_{q_X} H_*(F_q) \end{array}$$

commutes up to a chain homotopy that is natural in X .

3) *For $X \in \mathcal{S}_* \downarrow B$ consider the homological filtrations $\{L_n(p_X)\}$ and $\{L_n(q_X)\}$ of the complexes $C'_*(X) \otimes_{p_X} H_*(F_p)$ and, respectively, $C'_*(X) \otimes_{q_X} H_*(F_q)$ (4.7). The map g_X^∞ preserves these filtrations. Also, for every n the following diagram commutes:*

$$\begin{array}{ccc} L_n(p_X)/L_{n-1}(p_X) & \xrightarrow{g_X^\infty} & L_n(q_X)/L_{n-1}(q_X) \\ \cong \downarrow & & \downarrow \cong \\ C'_*(X) \otimes_{\pi_1 X} H_n(F_p) & \xrightarrow{\text{id} \otimes g|_{F_{p^*}}} & C'_*(X) \otimes_{\pi_1 X} H_n(F_q) \end{array}$$

The vertical isomorphisms in this diagram come from Proposition 4.8.

5. THE SECONDARY TRANSFER

We are now ready to give

Proof of Proposition 3.5. Consider the diagram (3-2). We want to construct a homotopy h_p between the maps $\lambda_E^h A(p^!)$ and $\chi(F_p) \lambda_B^h$. The combinatorial description of the linearization and the A -theory transfer given in Section 3 shows that the map $\lambda_E^h A(p^!)$ is induced by the functor

$$\Phi: \mathcal{R}^{fd}(B) \rightarrow \mathcal{Ch}^{fd}(\mathbb{Q})$$

that assigns to a retractive space X the relative chain complex $C_*(p^*X, E)$. On the other hand the map $\chi(F_p) \lambda_B^h$ is induced by the functor

$$\Psi: \mathcal{R}^{fd}(B) \rightarrow \mathcal{Ch}^{fd}(\mathbb{Q})$$

given by $\Psi(X) = C_*(X, B) \otimes H_*(F_p)$. We will build a sequence of intermediate functors joining Φ and Ψ .

First, for $X \in \mathcal{R}^{fd}(B)$ the inclusion map $i_X: B \hookrightarrow X$ induces an inclusion $\tilde{i}_X: E \hookrightarrow p^*X$. The naturality of the quasi-isomorphisms β_{p_X} described in Theorem 4.4 implies that we have a commutative diagram

$$\begin{array}{ccc} C_*(E) & \xrightarrow{\tilde{i}_{X*}} & C_*(p^*X) \\ \beta_p \downarrow & & \downarrow \beta_{p_X} \\ C'_*(B) \otimes_p H_*(F_p) & \xrightarrow{i_{X*} \otimes \text{id}} & C'_*(X) \otimes_{p_X} H_*(F_p) \end{array}$$

Define

$$C'_*(X, B) \otimes_{p_X} H_*(F_p) := \text{coker}(i_{X*} \otimes \text{id})$$

The assignment $X \mapsto C'_*(X, B) \otimes_{p_X} H_*(F_p)$ defines an almost exact functor

$$\Phi_1: \mathcal{R}^{fd} \rightarrow \mathcal{Ch}^{fd}(\mathbb{Q})$$

and the natural quasi-isomorphisms

$$\beta_{p_X}: C_*(p^*X, E) \xrightarrow{\simeq} C'_*(X, B) \otimes_{p_X} H_*(F_p)$$

define a natural transformation $\beta: \Phi \Rightarrow \Phi_1$. Denote by $K(\Phi_1): A(B) \rightarrow K(\mathbb{Q})$ the map induced by Φ_1 . The natural transformation β defines a homotopy

$$(5-1) \quad \lambda_E^h A(p^!) \simeq K(\Phi_1)$$

Next, since the map $i_{X*} \otimes \text{id}$ preserves the homological filtrations of the twisted tensor products we can define

$$L_n(p_X, p) := \text{coker}(L_n(p) \xrightarrow{i_{X*} \otimes \text{id}} L_n(p_X))$$

Proposition 4.8 shows that the filtration quotient $L_n(p_X, p)/L_{n+1}(p_X, p)$ can be identified with the chain complex

$$C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) := \text{coker}(C'_*(B) \otimes_{\pi_1 B} H_n(F_p) \rightarrow C'_*(X) \otimes_{\pi_1 X} H_n(F_p))$$

Notice that since F_p is a space of the homotopy finite type we have $H_q(F_p) = 0$ for q large enough, and so $\{L_n(p_X, p)\}$ is in fact a finite filtration. The assignments $X \rightarrow L_n(p_X, p)$, and $X \mapsto C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p)$ define almost exact functors $\mathcal{R}^{fd}(B) \rightarrow \mathcal{Ch}^{fd}(\mathbb{Q})$. These functors are connected by natural short exact sequences

$$(5-2) \quad 0 \rightarrow L_{n+1}(p_X, p) \rightarrow L_n(p_X, p) \rightarrow C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) \rightarrow 0$$

Let $\Phi_2: \mathcal{R}^{fd}(B) \rightarrow \mathcal{Ch}^{fd}(\mathbb{Q})$ denote the almost exact functor given by

$$\Phi_2(X) := \bigoplus_n C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p)$$

and let $K(\Phi_2): A(B) \rightarrow K(\mathbb{Q})$ be the map induced by Φ_2 . Applying repeatedly Waldhausen's additivity theorem to the sequences (5-2) we obtain a homotopy

$$(5-3) \quad K(\Phi_1) \simeq K(\Phi_2)$$

Assume now for a moment that $p: E \rightarrow B$ is a fibration with the trivial action of $\pi_1 B$ on $H_*(F)$. In this case the action of $\pi_1 X$ on $H_*(F_p)$ is trivial as well, so we have isomorphisms

$$\Phi_2(X) \cong C'_*(X, B) \otimes H_*(F_p)$$

Since by assumption X and B are path connected spaces we also have natural quasi-isomorphisms

$$C'_*(X, B) \otimes H_*(F_p) \simeq C_*(X, B) \otimes H_*(F_p) = \Psi(X)$$

As a consequence for every $X \in \mathcal{R}^{fd}(B)$ we obtain $\Phi_2(X) \simeq \Psi(X)$ which induces a homotopy

$$(5-4) \quad K(\Phi_2) \simeq \chi(F_p)\lambda_B^h$$

Concatenating the homotopies (5-1), (5-3), and (5-4) we get the desired homotopy η_p^h .

If p is an arbitrary unipotent fibration we need an additional step to pass between the maps $K(\Phi_2)$ and $\chi(F_p)\lambda_B^h$. In this case the action of $\pi_1 B$ need not be trivial, but we have a filtration $\{V^i\}$ of $H_*(F_p)$ where V^i is a $\pi_1 B$ -module and the action of $\pi_1 B$ on the quotients V^{i+1}/V^i is trivial. This defines a filtration $\{C'_*(X, B) \otimes_{\pi_1 X} V^i\}$ of the complex $\Phi_2(X)$. The quotients of this filtration are the (untwisted) tensor products $C'_*(X, B) \otimes (V^i/V^{i-1})$. Define a functor $\Phi_3: \mathcal{R}^{fd}(B) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$ by

$$\Phi_3(X) := \bigoplus_i C'_*(X, B) \otimes (V^i/V^{i-1})$$

Naturally, we also have a filtration $\{C'_*(X, B) \otimes V^i\}$ of the untwisted tensor product $C'_*(X, B) \otimes H_*(F_p)$ and $\Phi_3(X)$ is the direct sum of the quotients of this filtration. This means that using Waldhausen's additivity theorem (and the quasi-isomorphisms $C'_*(X, B) \otimes H_*(F_p) \simeq C_*(X, B) \otimes H_*(F_p)$) we get homotopies

$$(5-5) \quad K(\Phi_2) \simeq K(\Phi_3) \simeq \chi(F_p)\lambda_B^h$$

The homotopy η_p^h is then obtained as a concatenation of the homotopies (5-1), (5-3), and (5-5). \square

Proposition 3.5 and the arguments given in Section 3 complete the proof Theorem 1.2.

Let $C \in \mathcal{C}h^{fd}(\mathbb{Q})$ be a chain complex. Notice that our construction of $K(\mathbb{Q})$ lets us identify C with a point of $K(\mathbb{Q})$.

5.1. Definition. Let B be a path connected space and let $C \in \mathcal{C}h^{fd}(\mathbb{Q})$. Denote by $\text{Wh}_s^{\mathbb{Q}}(B)_C$ the homotopy fiber of the linearization map taken over the point $C \in K(\mathbb{Q})$.

$$\text{Wh}_s^{\mathbb{Q}}(B) := \text{hofib}(\lambda_B: Q(B_+) \rightarrow K(\mathbb{Q}))_C$$

If $0 \in \mathcal{C}h^{fd}(\mathbb{Q})$ is the zero chain complex we will denote $\text{Wh}_s^{\mathbb{Q}}(B) := \text{Wh}_s^{\mathbb{Q}}(B)_0$.

Let $p: E \rightarrow B$ be a unipotent bundle with a fiber F_p . By Theorem 1.2 for any $C \in \mathcal{C}h^{fd}(\mathbb{Q})$ we have a map

$$\mathrm{Wh}_s^{\mathbb{Q}}(B)_C \longrightarrow \mathrm{Wh}_s^{\mathbb{Q}}(E)_{C \otimes H_*(F_p)}$$

This gives rise to the following

5.2. Definition. The smooth secondary transfer of a unipotent bundle $p: E \rightarrow B$ is the map

$$\mathrm{Wh}_s^{\mathbb{Q}}(p'): \mathrm{Wh}_s^{\mathbb{Q}}(B) \longrightarrow \mathrm{Wh}_s^{\mathbb{Q}}(E)$$

determined by the Becker-Gottlieb transfer $Q(p')$ and the homotopy η_p given by Theorem 1.2.

It will be convenient to consider a variant of this definition in the setting of unipotent fibrations:

5.3. Definition. For a path connected space B let

$$\mathrm{Wh}_h^{\mathbb{Q}}(B) := (\lambda_B^h: A(B) \rightarrow K(\mathbb{Q}))_0$$

The homotopy secondary transfer of a unipotent fibration $p: E \rightarrow B$ is the map

$$\mathrm{Wh}_h^{\mathbb{Q}}(p'): \mathrm{Wh}_h^{\mathbb{Q}}(B) \longrightarrow \mathrm{Wh}_h^{\mathbb{Q}}(E)$$

determined by the transfer $A(p')$ and the homotopy η_p^h given by Proposition 3.5.

5.4. Note. Let $p: E \rightarrow B$ be a unipotent fibration with a fiber F_p . The construction of the homotopy η_p^h described in the proof of Proposition 3.5 makes use of a choice of a strong deformation retraction $H_*(F_p) \rightleftarrows C_*(F_p)$ and a choice of a unipotent filtration $\{V^i\}$ of $H_*(F_p)$. It is straightforward to check though that the homotopy class of the map $\mathrm{Wh}_h^{\mathbb{Q}}(p')$ is independent on these choices, and so it depends on the fibration p only. Likewise, if p is a unipotent bundle then the homotopy class of the smooth secondary transfer $\mathrm{Wh}_s^{\mathbb{Q}}(p')$ depends only on the bundle p .

6. HOMOTOPY INVARIANCE OF $\mathrm{Wh}_h^{\mathbb{Q}}(p')$

Let $f: E_1 \rightarrow E_2$ be a map of topological spaces. Such map defines an exact functor of Waldhausen categories

$$f_*: \mathcal{R}^{fd}(E_1) \rightarrow \mathcal{R}^{fd}(E_2)$$

given by $f_*(X) = X \cup_{E_1} E_2$ for $X \in \mathcal{R}^{fd}(E_1)$. This functor in turn induces a map $f_*: A(E_1) \rightarrow A(E_2)$. Consider the diagram

$$\begin{array}{ccc} A(E_1) & \xrightarrow{f_*} & A(E_2) \\ & \searrow \lambda_{E_1}^h & \swarrow \lambda_{E_2}^h \\ & K(\mathbb{Q}) & \end{array}$$

Recall that the map $\lambda_{E_1}^h$ is induced by the functor $\mathcal{R}^{fd}(E_1) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$ given by $X \mapsto C_*(X, E_1)$. Similarly, the map $\lambda_{E_2}^h f_*$ comes from the functor defined by $X \mapsto C_*(X \cup_{E_1} E_2, E_2)$. The natural quasi-isomorphisms $C_*(X, E_1) \rightarrow C_*(X \cup_{E_1} E_2, E_2)$ define a homotopy h_f between $\lambda_{E_1}^h$ and $\lambda_{E_2}^h f_*$. As a result we obtain a map

$$f_* : \text{Wh}_h^{\mathbb{Q}}(E_1) \longrightarrow \text{Wh}_h^{\mathbb{Q}}(E_2)$$

Our goal in this section is to prove the following

6.1. Proposition. *For $i = 1, 2$ let $p_i : E_i \rightarrow B$ be a unipotent fibration, and let $f : E_1 \rightarrow E_2$ be a fiberwise homotopy equivalence. There is a homotopy*

$$f_* \text{Wh}_h^{\mathbb{Q}}(p_1^!) \simeq \text{Wh}_h^{\mathbb{Q}}(p_2^!)$$

Proof. Let F_{p_i} denote the fiber of p_i . Using (11.2) we see that in order to obtain the desired homotopy it is enough to construct the following data:

- 1) a homotopy $H_f^A : A(B) \times I \rightarrow A(E_2)$ between $f_* A(p_1^!)$ and $A(p_2^!)$;
- 2) a homotopy $H_f^K : K(\mathbb{Q}) \times I \rightarrow K(\mathbb{Q})$ between the maps $\chi(F_{p_1})$ and $\chi(F_{p_2})$;
- 3) a homotopy of homotopies that fills the following diagram:

$$(6-1) \quad \begin{array}{ccc} \lambda_{E_2}^h f_* A(p_1^!) & \xrightarrow{\lambda_{E_2}^h H_f^A} & \lambda_{E_2}^h A(p_2^!) \\ \downarrow h_f \circ (A(p_1^!) \times \text{id}_I) & & \downarrow \eta_{p_2}^h \\ \lambda_{E_1}^h A(p_1^!) & & \\ \downarrow \eta_{p_1}^h & & \\ \chi(F_{p_1}) \lambda_B^h & \xrightarrow{H_f^K(\lambda_B^h \times \text{id}_I)} & \chi(F_{p_2}) \lambda_B^h \end{array}$$

Each vertex of this diagram represents a map $A(B) \rightarrow K(\mathbb{Q})$ and edges represent homotopies of such maps.

1) *Construction of H_f^A .* The map $f_* A(p_1^!)$ comes from the functor $\mathcal{R}^{fd}(B) \rightarrow \mathcal{R}^{fd}(E_2)$ given by $X \mapsto f_* p_1^* X$ while $A(p_2^!)$ is induced by the functor $X \mapsto p_2^* X$. Since f is a fiberwise homotopy equivalence the natural maps $f_* p_1^* X \rightarrow p_2^* X$ induced by f are weak equivalences, and so they define the homotopy H_f^A .

2) *Construction of H_f^K .* Recall that for $i = 1, 2$ the map $\chi(F_i)$ is induced by functor $\mathcal{C}h^{fd}(\mathbb{Q}) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$ given by $C \mapsto C \otimes H_*(F_i)$. Since the map $f|_{F_1} : F_1 \rightarrow F_2$ is a

homotopy equivalence it induces an isomorphism of homology groups of the fibers

$$(f|_{F_1})_* : H_*(F_1) \xrightarrow{\cong} H_*(F_2)$$

This gives a natural isomorphism of functors

$$- \otimes H_*(F_1) \Rightarrow - \otimes H_*(F_2)$$

The homotopy H_f^K is defined by this natural isomorphism.

3) *Construction of the homotopy of homotopies.* In order to show that the diagram (6-1) can be filled by a homotopy of homotopies we replace it first by the following diagram of functors:

$$\begin{array}{ccc}
 C_*(f_*p_1^*X, E_2) & \xrightarrow{\quad} & C_*(p_2^*X, E_2) \\
 \uparrow & \nearrow f_{X*} \text{ (1)} & \downarrow \beta_{p_{2X}} \\
 C_*(p_1^*X, E_1) & & \\
 \downarrow \beta_{p_{1X}} & \text{(2)} & \\
 C'_*(X, B) \otimes_{p_{1X}} H_*(F_{p_1}) & \xrightarrow{f_X^\infty} & C'_*(X, B) \otimes_{p_{2X}} H_*(F_{p_2}) \\
 \downarrow \text{additivity} & \text{(3)} & \downarrow \text{additivity} \\
 C'_*(X, B) \otimes H_*(F_{p_1}) & \xrightarrow{\text{id} \otimes (f|_{F_1})_*} & C'_*(X, B) \otimes H_*(F_{p_2})
 \end{array}$$

Each vertex of this diagram represents a functor $\mathcal{R}^{fd}(B) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$. The edges represent natural weak equivalences, with the exception of the lowest vertical edges where the passage between functors is obtained using additivity. The outer edges of this diagram correspond to the homotopies in the diagram (6-1). The maps $\beta_{p_{1X}}$ and $\beta_{p_{2X}}$ are the Brown quasi-isomorphisms (4.4) and the maps f_X^∞ come from Proposition 4.10.

In order to show that the diagram (6-1) can be filled by a homotopy of homotopies it is enough to show that each of the subdiagrams (1)-(3) in the above diagram of functors can be filled by a homotopy of homotopies. In the case of subdiagram (1) such a homotopy of homotopies exist since this subdiagram commutes. By Proposition 4.10 subdiagram (2) commutes up to a natural chain homotopy, so it again can be filled by a homotopy of homotopies. Proposition 4.10 says also that the maps f_X^∞ preserve the homological filtration of the twisted tensor products and that they induce the map $\text{id} \otimes (f|_{F_1})_*$ on the filtration quotients. This, together

with the fact that the map $(f|_{F_1})_*: H_*(F_1) \rightarrow H_*(F_2)$ is an isomorphism of $\pi_1 B$ -modules, implies that we also have a homotopy of homotopies filling subdiagram (3).

□

7. ADDITIVITY OF THE SECONDARY TRANSFER

Our goal of this section is to prove that secondary transfer maps have additivity properties that are analogous to additivity of the Becker-Gottlieb transfer and the A-theory transfer. We start by considering additivity of the homotopy secondary transfer:

7.1. Theorem. *For $i = 0, 1, 2$ let $p_i: E_i \rightarrow B$ be a unipotent fibration. Assume that we have maps of fibrations*

$$\begin{array}{ccccc} E_1 & \xleftarrow{j} & E_0 & \xrightarrow{\quad} & E_2 \\ & \searrow p_1 & \downarrow p_0 & \swarrow p_2 & \\ & & B & & \end{array}$$

where j is a cofibration over B . Let $E := E_1 \cup_{E_0} E_2$ and let $p: E \rightarrow B$ be the fibration given by $p := p_1 \cup_{p_0} p_2$. Then p is a unipotent fibration and we have

$$(7-1) \quad [\mathrm{Wh}_h^{\mathbb{Q}}(p')] = [k_{1*} \mathrm{Wh}_h^{\mathbb{Q}}(p_1)] + [k_{2*} \mathrm{Wh}_h^{\mathbb{Q}}(p_2)] - [k_{0*} \mathrm{Wh}_h^{\mathbb{Q}}(p_0)]$$

Here $k_{i*}: \mathrm{Wh}_h^{\mathbb{Q}}(E_i) \rightarrow \mathrm{Wh}_h^{\mathbb{Q}}(E)$ is induced by the map $k_i: E_i \rightarrow E$.

7.2. Lemma. *Consider a diagram of chain complexes*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow g' \\ A' & \xrightarrow{f'} & B' \end{array}$$

that commutes up to a chain homotopy h . There exists a map $\tilde{g}': \mathrm{Cyl}(f) \rightarrow B'$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Cyl}(f) \\ g \downarrow & & \downarrow \tilde{g}' \\ A' & \longrightarrow & B' \end{array}$$

commutes. Moreover \tilde{g}' is chain homotopic to the composition

$$\mathrm{Cyl}(f) \rightarrow B \xrightarrow{g'} B'$$

Proof. Recall that $\text{Cyl}(f)_n = A_n \oplus A_{n-1} \oplus B_n$. The map $\tilde{g}'_n: \text{Cyl}(f)_n \rightarrow B'_n$ is given by

$$\tilde{g}'_n(a_1, a_2, b) = f'g(a_1) + h(a_2) + g'(b)$$

The second statement of the lemma is easy to verify. \square

Proof of Theorem 7.1. The strategy of our proof is as follows. We will construct maps $f_1: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E_1)$ and $f_2: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E)$ such that $[k_1 * f_1] = [f_2]$. We will also show that

$$[\text{Wh}_h^{\mathbb{Q}}(p_1^!)] = [j_* \text{Wh}_h^{\mathbb{Q}}(p_0^!)] + [f_1] \quad \text{and} \quad [\text{Wh}_h^{\mathbb{Q}}(p^!)] = [k_{2*} \text{Wh}_h^{\mathbb{Q}}(p_2^!)] + [f_2]$$

Since $k_0 = k_1 j$ the first of these equations will give

$$[k_{1*} \text{Wh}_h^{\mathbb{Q}}(p_1^!)] = [k_{0*} \text{Wh}_h^{\mathbb{Q}}(p_0^!)] + [k_{1*} f_1] = [k_{0*} \text{Wh}_h^{\mathbb{Q}}(p_0^!)] + [f_2]$$

which, combined with the second equation, will yield the formula (7-1).

The construction of the map f_1 will proceed following the scheme outlined in (11.1). First, we will construct a map $f_1^A: A(B) \rightarrow A(E_1)$. Subsequently we will consider the diagram

$$(7-2) \quad \begin{array}{ccc} A(B) & \xrightarrow{f_1^A} & A(E_1) \\ \lambda_B \downarrow & & \downarrow \lambda_{E_1} \\ K(\mathbb{Q}) & \xrightarrow{\chi(\text{Cone}(j|_{F_0*}))} & K(\mathbb{Q}) \end{array}$$

Here $\text{Cone}(j|_{F_0*})$ denotes the mapping cone of the map $(j|_{F_0})_*: H_*(F_0) \rightarrow H_*(F_1)$, and the map

$$\chi(\text{Cone}(j|_{F_0*})): K(\mathbb{Q}) \rightarrow K(\mathbb{Q})$$

is induced by the functor $\mathcal{C}h^{fd}(\mathbb{Q}) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$ given by tensoring by $\text{Cone}(j|_{F_0*})$. We will show that the diagram (7-2) commutes up to a preferred homotopy h_1 . This homotopy together with the map f_1^A will define the map f_1 .

In order to obtain the map f_1^A recall that the map $j_* A(p_0^!)$ is induced by the functor that assigns to a space $X \in \mathcal{R}^{fd}(B)$ the space $j_* p_0^* X \in \mathcal{R}^{fd}(E_1)$, and that the map $A(p_1^!)$ is induced by the functor $X \mapsto p_1^* X$. For $X \in \mathcal{R}^{fd}(B)$ we have a cofibration $j_* p_0^* X \rightarrow p_1^* X$. Let $M_X \in \mathcal{R}^{fd}(E_1)$ denote the cofiber of this map. The assignment $X \mapsto M_X$ defines an exact functor $\mathcal{R}^{fd}(B) \rightarrow \mathcal{R}^{fd}(E_1)$. The map f_1^A is induced by this functor. We notice here that the above constructions gives a short exact sequence of functors

$$(7-3) \quad j_* p_0^* X \rightarrow p_1^* X \rightarrow M_X$$

Applying Waldhausen's additivity theorem we obtain a homotopy

$$(7-4) \quad A(p_1^!) \simeq j_* A(p_0^!) + f_1^A$$

Next, in order to describe a homotopy that fills the diagram (7-2) consider the following diagram of functors:

$$(7-5) \quad \begin{array}{ccccc} & C_*(j_{X*} p_0^* X, E_1) & & & C_*(M_X, E_1) \\ & \uparrow \simeq & \searrow & & \uparrow \simeq \\ & C_*(p_0^* X, E_0) & \xrightarrow{j_{X*}} & C_*(p_1^* X, E_1) & \longrightarrow & \text{coker}(j_{X*}) \\ & \uparrow = & & \uparrow & & \uparrow \\ & C_*(p_0^* X, E_0) & \longrightarrow & \text{Cyl}(j_{X*}) & \longrightarrow & \text{Cone}(j_{X*}) \\ & \downarrow \beta_{p_{0,X}} & & \downarrow g_X & & \downarrow \\ C'_*(X, B) \otimes_{p_{0,X}} H_*(F_0) & \longrightarrow & \text{Cyl}(j_X^\infty) & \longrightarrow & \text{Cone}(j_X^\infty) \\ & \downarrow \text{additivity} & \downarrow \text{additivity} & & \downarrow \text{additivity} \\ C_*(X, B) \otimes H_*(F_0) & \longrightarrow & C_*(X, B) \otimes \text{Cyl}(j|_{F_0*}) & \longrightarrow & C_*(X, B) \otimes \text{Cone}(j|_{F_0*}) \end{array}$$

The map j_{X*} in this diagram is induced by the map of fibrations $j_X: p_0^* X \rightarrow p_1^* X$. The complexes $\text{Cyl}(j_{X*})$ and $\text{Cone}(j_{X*})$ are, respectively, the mapping cylinder and the mapping cone of j_{X*} . Similarly $\text{Cyl}(j_X^\infty)$ and $\text{Cone}(j_X^\infty)$ are the mapping cylinder and the mapping cone of the map

$$j_X^\infty: C'_*(X, B) \otimes_{p_{0,X}} H_*(F_0) \rightarrow C'_*(X, B) \otimes_{p_{1,X}} H_*(F_1)$$

given by Proposition 4.10. Finally, $\text{Cyl}(j|_{F_0*})$ and $\text{Cone}(j|_{F_0*})$ are the mapping cylinder and the mapping cone of the map $j|_{F_0*}: H_*(F_0) \rightarrow H_*(F_1)$. The horizontal maps are defined in the obvious way so that each row of the diagram forms a short exact sequence.

All vertical maps are quasi-isomorphism. They are defined in the obvious way with the exception the map g_X which is given as follows. By Proposition 4.10 we have a diagram

$$\begin{array}{ccc} C_*(p_0^* X, E_0) & \xrightarrow{j_{X*}} & C_*(p_1^* X, E_1) \\ \downarrow \beta_{p_{0,X}} & & \downarrow \beta_{p_{1,X}} \\ C'_*(X, B) \otimes_{p_{0,X}} H_*(F_0) & \xrightarrow{j_X^\infty} & C'_*(X, B) \otimes_{p_{1,X}} H_*(F_1) \end{array}$$

that commutes up to a chain homotopy. Using Lemma 7.2 we obtain from here a commutative diagram

$$\begin{array}{ccc}
 C_*(p_0^*X, E_0) & \xrightarrow{j_{X*}} & \text{Cyl}(j_{X*}) \\
 \beta_{p_0, X} \downarrow & & \downarrow \bar{\beta}_{p_1, X} \\
 C'_*(X, B) \otimes_{p_0, X} H_*(F_0) & \xrightarrow{j_X^\infty} & C'_*(X, B) \otimes_{p_1, X} H_*(F_1)
 \end{array}$$

The map g_X is the composition of $\bar{\beta}_{p_1, X}$ and the inclusion

$$C'_*(X, B) \otimes_{p_1, X} H_*(F_1) \rightarrow \text{Cyl}(j_X^\infty)$$

The lowest vertical edges of in the diagram indicate a passage between chain complexes using additivity. This means the following construction. Since the map j_* preserves the homological filtrations on $C'_*(X, B) \otimes_{p_0, X} H_*(F_0)$ and $C'_*(X, B) \otimes_{p_1, X} H_*(F_1)$ the complexes $\text{Cyl}(j_X^\infty)$ and $\text{Cone}(j_X^\infty)$ are endowed with induced filtrations. Each lowest vertical edge indicates a passage from the filtered chain complex to the direct sum of the filtration quotients. As usual, after passage to the induced maps $A(B) \rightarrow K(\mathbb{Q})$ each additivity edge gives a homotopy obtained using Waldhausen's additivity theorem. We note here that the additivity edges are related to one another as follows. The maps in the short exact sequence

$$C'_*(X, B) \otimes_{p_0, X} H_*(F_0) \longrightarrow \text{Cyl}(j_X^\infty) \longrightarrow \text{Cone}(j_X^\infty)$$

preserve filtrations. Moreover, their restrictions to the filtration subcomplexes also form short exact sequences, and so do the induced maps of filtrations quotients. The bottom row of the diagram is the direct sum of these short exact sequences of these filtration quotients.

Each vertex of in the diagram (7-5) induces a map $A(B) \rightarrow K(\mathbb{Q})$. All vertical edges define homotopies of such maps. Concatenation of homotopies defined by rightmost vertical edges gives a homotopy filling the diagram (7-2). This homotopy, together with the map f_1^A defines the map $f_1 : \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E_1)$.

The existence of a homotopy between the maps $\text{Wh}_h^{\mathbb{Q}}(p_1^!)$ and $j_*\text{Wh}_h^{\mathbb{Q}}(p_0^!) + f_1$ follows directly from the above construction. The map $j_*\text{Wh}_h^{\mathbb{Q}}(p_0^!)$ is defined by the map $A(p_1^!)$ and the homotopy induced by the leftmost vertical edges in the diagram (7-5). The map $\text{Wh}_h^{\mathbb{Q}}(p_1^!)$ is homotopic to the map defined by $A(p_1^!)$ and the homotopy induced by the middle vertical edges in (7-5). As we have already noticed applying Waldhausen's additivity theorem to the short exact sequence of functors (7-3) defines a homotopy between $A(p_1^!)$ and $j_*A(p_0^!) + f_1^A$. In order to lift this homotopy to a homotopy between $\text{Wh}_h^{\mathbb{Q}}(p_1^!)$ and $j_*\text{Wh}_h^{\mathbb{Q}}(p_0^!) + f_1$ it is enough to apply the additivity theorem to the horizontal short exact sequences in the diagram (7-5).

Construction of the map $f_2: \text{Wh}_h^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(E)$ proceeds in exactly the same way as construction of f_1 , with the cofibration $k_2: E_2 \rightarrow E$ used in place of j . By the same argument as above we obtain a homotopy $\text{Wh}_h^{\mathbb{Q}}(p^!) \simeq k_* \text{Wh}_h^{\mathbb{Q}}(p_2^!) + f_2$. Finally, the fact that the maps $k_{1*}f_1$ and f_2 are homotopic can be verified directly by inspecting the construction of f_1 and f_2 \square

A statement analogous to Theorem 7.1 holds for the smooth secondary transfer. Given a smooth bundle $p: E \rightarrow B$ whose fibers are manifolds with a boundary by the vertical boundary of p we will understand the smooth bundle $\partial^v p: \partial^v E \rightarrow B$ obtained by restricting p to the union of boundaries of its fibers. We have:

7.3. Theorem. *Let $p: E \rightarrow B$ be a smooth bundle with closed fibers, and for $i = 0, 1, 2$ let $p_i: E_i \rightarrow B$ be unipotent subbundles of p such that p_0 is the vertical boundary of both p_1 and p_2 , and that $E = E_1 \cup_{E_0} E_2$. Then p is a unipotent bundle and we have*

$$[\text{Wh}_s^{\mathbb{Q}}(p^!)] = [k_{1*} \text{Wh}_s^{\mathbb{Q}}(p_1)] + [k_{2*} \text{Wh}_s^{\mathbb{Q}}(p_2)] - [k_{0*} \text{Wh}_s^{\mathbb{Q}}(p_0)]$$

Here $k_{i*}: \text{Wh}_s^{\mathbb{Q}}(E_i) \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$ is induced by the map $k_i: E_i \rightarrow E$.

The proof of this fact is similar to the proof of Theorem 7.1. The main difference is that instead of constructing homotopies of maps $A(B) \rightarrow A(E)$ such as the one given in (7-4) we need to construct analogous homotopies of maps $Q(B_+) \rightarrow Q(E_+)$. This can be done by arguments similar to these given in the proof of Theorem 7.1, but working in the categories of partitions (3.1) instead of categories of retractive spaces.

8. COMPOSITION OF UNIPOTENT FIBRATIONS

Our next goal is to give a proof of Theorem 1.3. Recall that this theorem states that the smooth secondary transfer preserves compositions of unipotent bundles. We will also show that an analogous property holds for the homotopy secondary transfer of unipotent fibrations:

8.1. Theorem. *If $p: E \rightarrow B$ and $q: D \rightarrow E$ are unipotent fibrations then*

$$\text{Wh}_h^{\mathbb{Q}}((pq)^!) \simeq \text{Wh}_h^{\mathbb{Q}}(q^!) \circ \text{Wh}_h^{\mathbb{Q}}(p^!)$$

The statements of Theorems 1.3 and 8.1 rely on the fact that the composition of unipotent bundles (or unipotent fibrations) is again unipotent. While this property is implicitly present in the work of Igusa [9] we give its proof below for completeness, and also because its main ingredient, Lemma 8.2, will be needed later on.

8.2. Lemma ([9, Lemma 8.9]). *Let*

$$F_p \rightarrow E \xrightarrow{p} B$$

be a unipotent fibration. There is a finite sequence of unipotent fibrations

$$\begin{array}{ccccccc} E_0 & \longrightarrow & E_1 & \longrightarrow & \dots & \longrightarrow & E_k \\ & \searrow p_0 & \downarrow p_1 & & & \nearrow p_k & \\ & & B & & & & \end{array}$$

such that

- (i) $E_0 = \Sigma_B^n E$ for some $n \geq 0$;
- (ii) $p_k: E_k \rightarrow B$ is a rational homotopy equivalence;
- (iii) for every i we have a cofibration sequence over B :

$$B \times S^{n_i} \xrightarrow{\alpha_i} E_i \longrightarrow E_{i+1}$$

(i.e. $E_{i+1} = E_i \cup_{B \times S^{n_i}} B \times D^{n_i+1}$).

We denote here by $\Sigma_B^n E \rightarrow B$ the n -fold fiberwise suspension of the fibration p , while $B \times S^{n_i} \rightarrow B$ and $B \times D^{n_i+1} \rightarrow B$ are, respectively, a product sphere bundle and a product disc bundle.

8.3. Theorem (Igusa). *If $q: D \rightarrow E$ and $p: E \rightarrow B$ are unipotent bundles (resp. unipotent fibrations) then the composition $pq: D \rightarrow B$ is also a unipotent bundle (resp. a unipotent fibration).*

Proof. Let F_p, F_q, F_{pq} denote the fibers of p, q, pq , respectively. Since the only non-trivial property we need to verify is that the action of $\pi_1 B$ on $H_*(F_{pq})$ is unipotent it is enough to show that the statement of the theorem holds for unipotent fibrations. We will split our argument into a few steps.

Step 1. The fibration pq is unipotent for an arbitrary unipotent fibration p and any product fibration $q: E \times F_q \rightarrow E$.

Indeed, in this case we have an isomorphism of $\pi_1 B$ -modules

$$H_*(F_{pq}) \cong H_*(F_p) \otimes H_*(F_q)$$

where the action of $\pi_1 B$ on the right hand side is given by $\alpha(x \otimes y) = \alpha x \otimes y$.

Step 2. The fibration pq is unipotent for an arbitrary unipotent fibration $p: E \rightarrow B$ and any fibration $q: D \rightarrow E$ with fiber F_q such that $\tilde{H}_*(F_q) = 0$.

This holds since the map $(q|_{F_{pq}})_*: H_*(F_{pq}) \rightarrow H_*(F_p)$ is in this case an isomorphism of $\pi_1 B$ -modules.

Step 3. For $i = 0, 1, 2$ let $p_i: E_i \rightarrow B$ be a fibration with a path connected base space B and fiber F_i of a finite homotopy type. Assume that we have maps of fibrations

$$E_1 \xleftarrow{f} E_0 \xrightarrow{i} E_2$$

where i is a cofibration over B . Let $p: E_1 \cup_{E_0} E_2 \rightarrow B$ be the pushout map. If three of the fibrations p_0, p_1, p_2, p are unipotent then so is the fourth.

This follows from the Mayer-Vietoris sequence for the homology of the fibers and the fact that unipotent $\pi_1 B$ -modules form a Serre category.

Step 4. As an application of Step 3 we obtain that if p, q are unipotent fibrations and $\Sigma_E q: \Sigma_E D \rightarrow E$ is a fiberwise suspension of q then pq is a unipotent fibration iff $p\Sigma_E q$ is unipotent.

Step 5. Assume now that p and q are arbitrary unipotent fibrations. Applying Lemma 8.2 to q we obtain a sequence of fibrations

$$\begin{array}{ccccccc} D_0 & \longrightarrow & D_1 & \longrightarrow & \dots & \longrightarrow & D_k \\ & \searrow q_0 & \downarrow q_1 & & & \nearrow q_k & \\ & & E & & & & \end{array}$$

We will show that pq_i is a unipotent fibration for all i . In case of $i = k$ this is a consequence of Step 2. Arguing inductively, assume that pq_{i+1} is unipotent for some i . Since pq_{i+1} is the pushout in the diagram of fibrations over B

$$E \times D^{n_i+1} \leftarrow E \times S^{n_i} \rightarrow D_i$$

and the fibrations $E \times D^{n_i+1} \rightarrow B, E \times S^{n_i} \rightarrow B$ are unipotent by Step 1, we obtain using Step 3 that pq_i is also a unipotent fibration. As a consequence we get that pq_0 is a unipotent fibration. Since q_0 is an iterated fiberwise suspension of q by Step 4 we obtain that pq is unipotent. \square

9. COMPOSITION OF SECONDARY TRANSFERS

In this section we prove Theorems 1.3 and 8.1. Our strategy will be as follows. First, in (9.1) and (9.2) we show that Theorem 8.1 holds in two special cases. Then we will use an argument of Igusa to show that the general statement of (8.1) follows from these special cases. Finally, we will show that essentially the same reasoning can be used to obtain a proof of Theorem 1.3.

9.1. Lemma. *Let $p: E \rightarrow B$, $q: D \rightarrow E$ be unipotent fibrations with fibers F_p and F_q respectively. If $\tilde{H}_*(F_q) = 0$ then*

$$\mathrm{Wh}_h^{\mathbb{Q}}((pq)^!) \simeq \mathrm{Wh}_h^{\mathbb{Q}}(q^!) \circ \mathrm{Wh}_h^{\mathbb{Q}}(p^!)$$

Proof. Let F_{pq} denote the fiber of pq . Recall (5.3) that the map $\mathrm{Wh}_h^{\mathbb{Q}}((pq)^!)$ is defined by the diagram

$$\begin{array}{ccc} A(B) & \xrightarrow{A((pq)^!)} & A(D) \\ \lambda_B^h \downarrow & & \downarrow \lambda_D^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_{pq})} & K(\mathbb{Q}) \end{array}$$

which commutes up to the homotopy η_{pq}^h . On the other hand, the composition $\mathrm{Wh}_h^{\mathbb{Q}}(q^!)\mathrm{Wh}_h^{\mathbb{Q}}(p^!)$ is induced by the diagram

$$\begin{array}{ccccc} A(B) & \xrightarrow{A(p^!)} & A(E) & \xrightarrow{A(q^!)} & A(D) \\ \lambda_B^h \downarrow & & \downarrow \lambda_E^h & & \downarrow \lambda_D^h \\ K(\mathbb{Q}) & \xrightarrow{\chi(F_p)} & K(\mathbb{Q}) & \xrightarrow{\chi(F_q)} & K(\mathbb{Q}) \end{array}$$

The left square in this diagram commutes up to the homotopy η_p^h , and the right square commutes up to the homotopy η_q^h . It follows that the outer square, defining the map $\mathrm{Wh}_h^{\mathbb{Q}}(q^!)\mathrm{Wh}_h^{\mathbb{Q}}(p^!)$, commutes up to the homotopy obtained by concatenating $\eta_q \circ (A(p^!) \times \mathrm{id}_I)$ with $\chi(F_q)\eta_p$.

By (11.2) in order to obtain a homotopy between $\mathrm{Wh}_h^{\mathbb{Q}}(q^!)\mathrm{Wh}_h^{\mathbb{Q}}(p^!)$ and $\mathrm{Wh}_h^{\mathbb{Q}}((pq)^!)$ it is enough to construct the following data:

- 1) a homotopy $H_A(p, q): A(B) \times I \rightarrow A(D)$ between $A((pq)^!)$ and $A(q^!)A(p^!)$;
- 2) a homotopy $H_K: K(\mathbb{Q}) \times I \rightarrow K(\mathbb{Q})$ between the maps $\chi(F_{pq})$ and $\chi(F_q)\chi(F_p)$;

3) a homotopy of homotopies that fills the following diagram:

$$(9-1) \quad \begin{array}{ccc} \lambda_D^h A(q^!) A(p^!) & \xrightarrow{\lambda_D^h H_A(p, q)} & \lambda_D^h A((pq)^!) \\ \eta_q^h \circ (A(p^!) \times \text{id}_I) \downarrow & & \downarrow \eta_{pq}^h \\ \chi(F_q) \lambda_E^h A(p^!) & & \\ \chi(F_q) \eta_p^h \downarrow & & \\ \chi(F_q) \chi(F_p) \lambda_B^h & \xrightarrow{H_K(\lambda_B^h \times \text{id}_I)} & \chi(F_{pq}) \lambda_B^h \end{array}$$

Each vertex of this diagram represents a map $A(B) \rightarrow K(\mathbb{Q})$ and edges represent homotopies of such maps.

1) *Construction of $H_A(p, q)$.* The map $A((pq)^!)$ comes from the functor $\mathcal{R}^{fd}(B) \rightarrow \mathcal{R}^{fd}(D)$ that assigns to a retractive space X the space $(pq)^* X$, while the composition $A(q^!) A(p^!)$ comes from the functor that sends X to $q^* p^* X$. The canonical isomorphisms

$$(pq)^* X \xrightarrow{\cong} q^* p^* X$$

define a natural transformation of functors, and so they induce a homotopy between $A((pq)^!)$ and $A(q^!) A(p^!)$. This is the homotopy $H_A(p, q)$.

2) *Construction of H_K .* Recall that the maps $\chi(F_p), \chi(F_q)$ and $\chi(F_{pq})$ are induced by functors $\mathcal{C}h^{fd}(\mathbb{Q}) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$ than tensor a chain complex C by, respectively, $H_*(F_p), H_*(F_q)$, and $H_*(F_{pq})$. As a consequence the map $\chi(F_q) \chi(F_p)$ is induced by the functor that tensors $C \in \mathcal{C}h^{fd}(\mathbb{Q})$ by the chain complex $H_*(F_p) \otimes H_*(F_q)$. Since $\tilde{H}_*(F_q) = 0$ we have an isomorphism

$$H_*(F_{pq}) \xrightarrow{(q|_{F_q})^*} H_*(F_p) \xrightarrow{\cong} H_*(F_p) \otimes H_*(F_q)$$

This induces a natural isomorphism of functors

$$- \otimes H_*(F_{pq}) \Rightarrow - \otimes (H_*(F_p) \otimes H_*(F_q))$$

which, in turn, defines the homotopy H_K .

3) *Construction of the homotopy of homotopies.* In order to show that the diagram (9-1) can be filled by a homotopy of homotopies we will first replace it by the

underlying diagram of functors:

$$\begin{array}{ccccc}
 C_*(q^*p^*X, D) & \xleftarrow{\quad \cong \quad} & C_*((pq)^*X, D) \\
 \downarrow \beta_q & \searrow q_* & \swarrow q_* & \downarrow \beta_{pq} \\
 C'_*(p^*X, E) \otimes H_*(F_q) & & & \\
 \downarrow \cong & \textcircled{1} & & \\
 C_*(p^*X, E) \otimes H_*(F_q) & \xleftarrow{\quad \cong \quad} & C_*(p^*X, E) & \textcircled{4} \\
 \downarrow \beta_q \otimes \text{id} & \textcircled{2} & \downarrow \beta_q & \\
 C'_*(X, B) \otimes_{p_X} H_*(F_p) \otimes H_*(F_q) & \xleftarrow{\quad \cong \quad} & C'_*(X, B) \otimes_{p_X} H_*(F_p) & \xleftarrow{q^\infty} C'_*(X, B) \otimes_{(pq)_X} H_*(F_{pq}) \\
 \downarrow \text{additivity} & \textcircled{3} & \downarrow \beta_q & \downarrow \text{additivity} \\
 C_*(X, B) \otimes H_*(F_p) \otimes H_*(F_q) & \xleftarrow{\quad \cong \quad} & C_*(X, B) \otimes H_*(F_p) & \xleftarrow{\text{id} \otimes q_*} C_*(X, B) \otimes H_*(F_{pq}) \\
 & \textcircled{5} & &
 \end{array}$$

Each vertex of this diagram represents a functor $\mathcal{R}^{fd}(B) \rightarrow \mathcal{C}h^{fd}(\mathbb{Q})$. The edges represent natural weak equivalences, with the exception of the lowest vertical edges where the passage between functors is obtained using additivity. The outer edges of this diagram correspond to the homotopies in the diagram (9-1). Since $\tilde{H}_*(F_q) = 0$ the twisted tensor product of the fibration $q^*p^*X \rightarrow p^*X$ is just the untwisted tensor product $C_*(p^*X, E) \otimes H_*(F_q)$. In effect the homotopy $\eta_q \circ (A(p^!) \times \text{id}_I)$ in (9-1) is induced simply by the the quasi-isomorphisms

$$C_*(q^*p^*X, D) \xrightarrow{\beta_q} C'_*(p^*X, E) \otimes H_*(F_q) \xrightarrow{\cong} C_*(p^*X, E) \otimes H_*(F_q)$$

without using additivity.

In order to show that the diagram (9-1) can be filled by a homotopy of homotopies it is enough to show that each of the subdiagrams in the above diagram of functors can be filled by a homotopy of homotopies. In the case of the subdiagrams (1)–(4) such homotopies of homotopies exist since subdiagrams (1) and (3) commute strictly and subdiagrams (2) and (4) commute up to natural chain homotopies. Homotopy commutativity of the subdiagram (2) follows from [4, (7.4)]. The subdiagram (4) is homotopy commutative since it is obtained by applying Proposition

4.10 to the map of fibrations

$$\begin{array}{ccc} D & \xrightarrow{q} & E \\ & \searrow pq \quad \swarrow p & \\ & B & \end{array}$$

Finally, the subdiagram (5) can be filled by a homotopy of homotopies since the maps

$$C_*(X, B) \otimes_{(pq)_X} H_*(F_{pq}) \xrightarrow{q^\infty} C_*(X, B) \otimes_{p_X} H_*(F_p) \xrightarrow{\cong} C_*(X, B) \otimes_{p_X} H_*(F_p) \otimes H_*(F_q)$$

preserve the homological filtrations and induce isomorphisms on the filtration quotients. \square

9.2. Lemma. *Let $p: E \rightarrow B$ be a unipotent fibration and let $q: E \times S^0 \rightarrow E$ be the product fibration with fiber S^0 . Then*

$$\mathrm{Wh}_h^{\mathbb{Q}}((pq)^!) \simeq \mathrm{Wh}_h^{\mathbb{Q}}(q^!) \circ \mathrm{Wh}_h^{\mathbb{Q}}(p^!)$$

Proof. The basic outline of our argument is the same as in the proof of Lemma 9.1. The same construction as in the proof of that lemma gives a homotopy $H_A(p, q)$ between the maps $A((pq)^!)$ and $A(p^!)A(q^!)$. Next, we need a homotopy H_K between the maps $\chi(F_p \times S^0)$ and $\chi(S^0)\chi(F_p)$. The first of these maps comes from the functor $C \mapsto C \otimes H_*(F_p \times S^0)$ while the second map is induced by the functor $C \mapsto C \otimes H_*(F_p) \otimes H_*(S^0)$. The isomorphism $H(F_p) \otimes H_*(S^0) \cong H_*(F_p \times S^0)$ induces a natural isomorphism of the above functors, which in turn defines the homotopy H_K .

As the result of these constructions we obtain a diagram of homotopies (9-1) (with $F_q = S^0$ and $F_{pq} = F_p \times S^0$). The remaining step is to show that this diagram can be filled by a homotopy of homotopies. The existence of such a homotopy of homotopies can be verified in a straightforward manner using the fact that for $X \in \mathcal{R}^{fd}(B)$ we have a commutative diagrams:

$$\begin{array}{ccc} C_*(q^*p^*X, E \times S^0) & \xrightarrow{\cong} & C(p^*X, E) \otimes H_*(S^0) \\ \beta_{pq} \downarrow & & \downarrow \beta_p \otimes \mathrm{id} \\ C'_*(X, B) \otimes_{(pq)_X} H(E \times S^0) & \xrightarrow{\cong} & (C(X, B) \otimes_{p_X} H_*(F_p)) \otimes H_*(S^0) \end{array}$$

\square

9.3. Igusa's argument. In [9] Igusa used Lemma 8.2 to show that two higher torsion invariants of unipotent bundles coincide. We will adapt this argument to prove Theorems 1.3 and 8.1. The main idea of Igusa's proof is to produce a “difference torsion”, which is new invariant that measures the difference between the given two invariants of bundles, and then to show that this difference torsion vanishes for all

unipotent bundles. The essential properties of Igusa's difference torsion are encapsulated in Definition 9.4. Proposition 9.5 spells out the conditions that guarantee its vanishing.

9.4. Definition. Let B be a space of the homotopy type of a finite CW-complex, and let Λ be an abelian group. An additive homotopy B -invariant of unipotent fibrations with values in Λ is an assignment Φ that associates to each unipotent fibration $p: E \rightarrow B$ an element $\Phi(p) \in \Lambda$ and that satisfies the following conditions:

(**additivity**) given maps of fibrations over B

$$\begin{array}{ccccc} E_1 & \xleftarrow{j} & E_0 & \xrightarrow{\quad} & E_2 \\ & \searrow p_1 & \downarrow p_0 & \swarrow p_2 & \\ & & B & & \end{array}$$

where p_i is a unipotent fibration for $i = 0, 1, 2$, and j is a cofibration we have

$$\Phi(p_1 \cup_{p_0} p_2) = \Phi(p_1) + \Phi(p_2) - \Phi(p_0)$$

(**homotopy invariance**) if unipotent fibrations $p_i: E_i \rightarrow B$ ($i = 1, 2$) are fiberwise homotopy equivalent then $\Phi(p_1) = \Phi(p_2)$.

9.5. Proposition. Let Φ in an additive homotopy B -invariant with values in Λ . Assume that

- $\Phi(p) = 0$ if $p: B \times S^0 \rightarrow B$ is a product fibration;
- $\Phi(p) = 0$ if the map $p: E \rightarrow B$ is a rational homotopy equivalence.

Then $\Phi(p) = 0$ for all unipotent fibrations $p: E \rightarrow B$.

Proof. First notice that $\Phi(p) = 0$ if $p: B \times F \rightarrow B$ is a product fibration where F is either a disc or a sphere. Indeed, in the first case p is a rational homotopy equivalence. If F is a sphere we can argue inductively starting with $F = S^0$ and using additivity.

Next, let $p: E \rightarrow B$ be a unipotent fibration. Applying Lemma 8.2 to p we obtain a sequence of fibrations

$$(9-2) \quad \begin{array}{ccccccc} E_0 & \longrightarrow & E_1 & \longrightarrow & \dots & \longrightarrow & E_k \\ & \searrow p_0 & \downarrow p_1 & \swarrow p_k & & & \\ & & B & & & & \end{array}$$

Since p_k is a rational homotopy equivalence we have $\Phi(p_k) = 0$. Next, since p_i is obtained as a pushout of p_{i-1} and product fibrations with disc and sphere fibers we can use additivity of Φ to get that $\Phi(p_i) = \Phi(p_{i-1})$. This gives

$$0 = \Phi(p_k) = \Phi(p_{k-1}) = \cdots = \Phi(p_0)$$

Finally, since p_0 is an n -fold fiberwise suspension of p we can use additivity of Φ again to get

$$\Phi(p) = (-1)^n \Phi(p_0) = 0$$

□

9.6. Note. Igusa uses a slightly different variant of Proposition 9.5 that will be also useful to us later on. Namely, for a smooth compact manifold B consider an assignment Φ that satisfies additivity and homotopy invariance properties as in Definition 9.4, but is defined only for unipotent bundles over B . Then the statement of Proposition 9.5 still holds: if Φ vanishes on the product bundle $B \times S^0 \rightarrow B$ and on bundles that are given by a rational homotopy equivalence then it vanishes on all unipotent bundles. The proof of this fact is essentially the same as the proof of Proposition 9.5 with two additional observations:

- the homotopy invariance of Φ lets us define this invariant on all smoothable unipotent fibrations over B , i.e. all fibrations that are fiberwise homotopy equivalent to a unipotent bundle.
- if $p: E \rightarrow B$ is a unipotent bundle then all fibrations appearing in the diagram (9-2) are smoothable [9, Lemma 8.6].

We are now ready to give

Proof of Theorem 8.1. Let $p: E \rightarrow B$, $q: D \rightarrow E$ be unipotent fibrations. We have commutative diagrams

$$\begin{array}{ccc} \mathrm{Wh}_h^{\mathbb{Q}}(B) & \xrightarrow{\mathrm{Wh}_h^{\mathbb{Q}}((pq)^!)} & \mathrm{Wh}_h^{\mathbb{Q}}(D) \\ i_B \downarrow & & \downarrow i_D \\ A(B) & \xrightarrow{A((pq)^!)} & A(D) \end{array} \quad \begin{array}{ccc} \mathrm{Wh}_h^{\mathbb{Q}}(B) & \xrightarrow{\mathrm{Wh}_h^{\mathbb{Q}}(q')\mathrm{Wh}_h^{\mathbb{Q}}(p')} & \mathrm{Wh}_h^{\mathbb{Q}}(D) \\ i_B \downarrow & & \downarrow i_D \\ A(B) & \xrightarrow{A(q')A(p')} & A(D) \end{array}$$

By the same argument as in the proof of Lemma 9.1 we can construct a homotopy $H_A(p, q)$ between the maps $A((pq)^!)$ and $A(q')A(p')$. By (11.3) this data defines a map $\varphi(p, q): \mathrm{Wh}_h^{\mathbb{Q}}(B) \rightarrow \Omega K(\mathbb{Q})$ such that $[\varphi(p, q)] = 0$ iff H_A admits a lift to a homotopy between $\mathrm{Wh}_h^{\mathbb{Q}}((pq)^!)$ and $\mathrm{Wh}_h^{\mathbb{Q}}(q')\mathrm{Wh}_h^{\mathbb{Q}}(p')$.

Fix the fibration $p: E \rightarrow B$. Let Φ_p be the assignment that associates to a unipotent fibration $q: D \rightarrow E$ the homotopy class $[\varphi(p, q)]$. We claim that Φ_p is an additive

homotopy E -invariant with values in the group $[\mathrm{Wh}_h^{\mathbb{Q}}(B), \Omega K(\mathbb{Q})]$ (9.4). In order to verify homotopy invariance of Φ_p assume that we have a fiberwise homotopy equivalence

$$\begin{array}{ccc} D_1 & \xrightarrow{f} & D_2 \\ & \searrow q_1 & \swarrow q_2 \\ & E & \end{array}$$

We need to check that the maps $\varphi(p, q_1)$ and $\varphi(p, q_2)$ are homotopic. By Proposition 6.1 we can construct a homotopy

$$H_f^{\mathrm{Wh}(E)}: \mathrm{Wh}_h^{\mathbb{Q}}(E) \times I \rightarrow \mathrm{Wh}_h^{\mathbb{Q}}(D_2)$$

between the maps $f_*\mathrm{Wh}_h^{\mathbb{Q}}(q_1^!)$ and $\mathrm{Wh}_h^{\mathbb{Q}}(q_2^!)$. As a consequence the map

$$\bar{H}_f^{\mathrm{Wh}(E)} := H_f^{\mathrm{Wh}(E)} \circ (\mathrm{Wh}_h^{\mathbb{Q}}(p^!) \times \mathrm{id}_I)$$

is a homotopy between $f_*\mathrm{Wh}_h^{\mathbb{Q}}(q_1^!)\mathrm{Wh}_h^{\mathbb{Q}}(p^!)$ and $\mathrm{Wh}_h^{\mathbb{Q}}(q_2^!)\mathrm{Wh}_h^{\mathbb{Q}}(p^!)$. On the other hand f is also a fiberwise homotopy equivalence of the fibrations pq_1 and pq_2 , so we have a homotopy

$$H_f^{\mathrm{Wh}(B)}: \mathrm{Wh}_h^{\mathbb{Q}}(B) \times I \rightarrow \mathrm{Wh}_h^{\mathbb{Q}}(D_2)$$

between the maps $f_*\mathrm{Wh}_h^{\mathbb{Q}}((pq_1)^!)$ and $\mathrm{Wh}_h^{\mathbb{Q}}((pq_2)^!)$. By the proof of Proposition 6.1 we also have a homotopy

$$H_f^{A(B)}: A(B) \times I \rightarrow A(D_2)$$

between the maps $f_*A((pq_1)^!)$ and $A((pq_2)^!)$ as well as a homotopy

$$\bar{H}_f^{A(E)} := H_f^{A(E)} \circ (A(p^!) \times \mathrm{id}_I)$$

between $f_*A(q_1^!)A(p^!)$ and $A(q_2^!)A(p^!)$. All these homotopies fit into commutative diagrams

$$\begin{array}{ccc} \mathrm{Wh}_h^{\mathbb{Q}}(B) \times I & \xrightarrow{H_f^{\mathrm{Wh}(B)}} & \mathrm{Wh}_h^{\mathbb{Q}}(D_2) \\ \downarrow i_B \times \mathrm{id}_I & & \downarrow i_{D_2} \\ A(B) \times I & \xrightarrow{H_f^{A(B)}} & A(D_2) \end{array} \quad \begin{array}{ccc} \mathrm{Wh}_h^{\mathbb{Q}}(B) \times I & \xrightarrow{\bar{H}_f^{\mathrm{Wh}(E)}} & \mathrm{Wh}_h^{\mathbb{Q}}(D) \\ \downarrow i_B \times \mathrm{id}_I & & \downarrow i_{D_2} \\ A(B) \times I & \xrightarrow{\bar{H}_f^{A(E)}} & A(D) \end{array}$$

Consider the diagram

$$(9-3) \quad \begin{array}{ccc} f_*A((pq_1)^!) & \xrightarrow{f_*H_A(p, q_1)} & f_*A(q_1^!)A(p^!) \\ \downarrow H_f^{A(B)} & & \downarrow \bar{H}_f^{A(E)} \\ A((pq_2)^!) & \xrightarrow{H_A(p, q_2)} & A(q_2^!)A(p^!) \end{array}$$

Each vertex of this diagram represents a map $A(B) \rightarrow A(D_2)$ and edges represent homotopies of such maps. This diagram is induced by diagram of functors $\mathcal{R}^{fd}(B) \rightarrow \mathcal{R}^{fd}(D_2)$ and natural equivalences of such functors. It is straightforward to check that this underlying diagram of functors commutes. This implies that the diagram (9-3) can be filled by a homotopy of homotopies. This homotopy of homotopies can be interpreted as a homotopy between $H_f^{A(B)}$ and $\bar{H}_f^{A(E)}$. By (11.3) this homotopy defines a map $\text{Wh}_h^{\mathbb{Q}}(B) \times I \rightarrow \Omega K(\mathbb{Q})$. One can check that this map determines a homotopy between $\varphi(p, q_1)$ and $\varphi(p, q_2)$. Additivity of Φ_p can be verified in a similar way, using additivity of secondary transfers.

By Lemma 9.1 and Lemma 9.2 we have $\Phi_p(q) = 0$ if q is a product fibration or a rational homotopy equivalence. Proposition 9.5 implies then that $[\varphi(p, q)] = \Phi_p(q) = 0$ for any unipotent fibration $q: D \rightarrow E$. \square

Proof of Theorem 1.3. Let $p: E \rightarrow B$ and $q: D \rightarrow E$ be unipotent bundles. The combinatorial construction of the Becker-Gottlieb transfer given in [2, 4.2] readily shows that there is homotopy $H_Q(p, q)$ between the maps $Q(q^!)Q(p^!)$ and $Q((pq)^!)$ such that we have a commutative diagram

$$(9-4) \quad \begin{array}{ccc} Q(B_+) \times I & \xrightarrow{H_Q(p, q)} & Q(D_+) \\ a_B \times \text{id}_I \downarrow & & \downarrow a_D \\ A(B) \times I & \xrightarrow{H_A(p, q)} & A(D) \end{array}$$

where $H_A(p, q)$ is the homotopy between $A(q^!)A(p^!)$ and $A((pq)^!)$ constructed in the proof of Theorem 8.1.

By (11.3) the maps $\text{Wh}_s^{\mathbb{Q}}(q^!)\text{Wh}_s^{\mathbb{Q}}(p^!)$, $\text{Wh}_s^{\mathbb{Q}}((pq)^!)$ and the homotopy H_Q define a map $\psi(p, q): \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \Omega K(\mathbb{Q})$ such that $\text{Wh}_s^{\mathbb{Q}}(q^!)\text{Wh}_s^{\mathbb{Q}}(p^!) \simeq \text{Wh}_h^{\mathbb{Q}}((pq)^!)$ if $[\psi(p, q)] = 0$.

It remains to show that $[\psi(p, q)] = 0$ for all unipotent bundles p, q . Consider the map $b_B: \text{Wh}_s^{\mathbb{Q}}(B) \rightarrow \text{Wh}_h^{\mathbb{Q}}(B)$ induced by the map of fibrations:

$$\begin{array}{ccc} \text{Wh}_s^{\mathbb{Q}}(B) & \xrightarrow{b_B} & \text{Wh}_h^{\mathbb{Q}}(B) \\ \downarrow & & \downarrow \\ Q(B_+) & \xrightarrow{a_B} & A(B) \\ \searrow \lambda_B & & \swarrow \lambda_B^h \\ & K(\mathbb{Q}) & \end{array}$$

The construction of the map $\psi(p, q)$ described in (11.3) combined with the commutativity of the diagram (9-4), gives

$$\psi(p, q) \simeq \varphi(p, q)b_B$$

where $\varphi(p, q)$ is the map defined in the proof of Theorem 8.1. In that proof we showed that $[\varphi(p, q)] = 0$, so also $[\psi(p, q)] = 0$. \square

10. SECONDARY TRANSFER AND SMOOTH TORSION

In [2] and [1] (joint with B. Williams and J. Klein) we described a homotopy theoretical construction of the smooth torsion of unipotent bundles and showed that it defines characteristic classes which coincide with the higher torsion invariants of Igusa and Klein. The construction of the smooth torsion of a bundle $p: B \rightarrow E$ proceeds as follows. Let $\eta_B: B \rightarrow Q(B_+)$ denote the coaugmentation map. By [2, Theorem 6.7] the map

$$\lambda_E Q(p^!) \eta_B: B \rightarrow K(\mathbb{Q})$$

is homotopic via a preferred homotopy ω_p to the constant map. This defines a map $\tau^s(p): B \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$ which is a lift of $Q(p^!) \eta_B$. The map $\tau^s(p)$ is the smooth torsion of the bundle p .

The secondary transfer of unipotent bundles described in this paper can be used to construct another map $\bar{\tau}^s(p): B \rightarrow \text{Wh}_s^{\mathbb{Q}}(E)$. Namely, since the identity map $\text{id}_B: B \rightarrow B$ can be considered as a unipotent bundle, it defines the smooth torsion $\tau^s(\text{id}_B): B \rightarrow \text{Wh}_s^{\mathbb{Q}}(B)$. We set

$$\bar{\tau}^s(p) := \text{Wh}_s^{\mathbb{Q}}(p^!) \tau^s(\text{id}_B)$$

Our final goal in this paper is to prove Theorem 1.4 which says that for any unipotent bundle p the maps $\tau^s(p)$ and $\bar{\tau}^s(p)$ are homotopic. We will also show that as a consequence the statement of Theorem 1.1 holds: for any pair of composable unipotent bundles p and q the higher torsion cohomology classes of p , q and pq are related by the formula (1-1).

The proof of Theorem 1.4 will use the same scheme as the proof of the composition formula of unipotent fibrations (Proposition 8.1). We will first show that this theorem holds when p is either a product bundle or it is a rational homotopy equivalence, and then we will use Igusa's argument (9.3) to extend this result to all unipotent bundles.

10.1. Lemma. *Let $p: E \rightarrow B$ be a unipotent bundle. If p is either the product bundle with fiber S^0 or a rational homotopy equivalence then $\tau^s(p) \simeq \bar{\tau}^s(p)$.*

Proof. This follows essentially from [1, §6]. We proved there that if $p: E \rightarrow B$ is a unipotent bundle that satisfies the assumptions of the Leray-Hirsch theorem then

the the quasi-isomorphisms

$$C_*(p^*X) \xrightarrow{\simeq} C_*(X) \otimes H_*(F_p)$$

given for $X \in \mathcal{R}^{fd}(B)$ by that theorem define a map

$$\mathrm{Wh}_{LH}^{\mathbb{Q}}(p^!): \mathrm{Wh}_s^{\mathbb{Q}}(B) \rightarrow \mathrm{Wh}_s^{\mathbb{Q}}(E)$$

and that $\tau^s(p) \simeq \mathrm{Wh}_{LH}^{\mathbb{Q}}(p^!)\tau^s(\mathrm{id}_B)$. It is straightforward to check that if p is either a product bundle $B \times S^0 \rightarrow B$ or a rational homotopy equivalence then $\mathrm{Wh}_{LH}^{\mathbb{Q}}(p^!) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(p^!)$. \square

Proof of Theorem 1.4. Both $\tau^s(p)$ and $\bar{\tau}^s(p)$ are defined as lifts of the map

$$\mathcal{Q}(p^!)\eta_B: B \rightarrow \mathcal{Q}(E_+)$$

As a consequence they define a map $\varrho(p): B \rightarrow \Omega K(\mathbb{Q})$ such that the homotopy class of the composition

$$B \xrightarrow{\varrho(p)} \Omega K(\mathbb{Q}) \longrightarrow \mathrm{Wh}_s^{\mathbb{Q}}(E)$$

coincides with the element $[\tau^s(p)] - [\bar{\tau}^s(p)] \in [B, \mathrm{Wh}_s^{\mathbb{Q}}(E)]$. It suffices to show that $[\varrho(p)] = 0$ in $[B, \Omega K(\mathbb{Q})]$.

We claim that the assignment $p \mapsto [\varrho(p)]$ is an additive homotopy B -invariant of unipotent bundles with values in $[B, \Omega K(\mathbb{Q})]$ (9.6). Indeed, additivity of this assignment follows essentially from the additivity of the secondary transfer (7.3) and the additivity of the smooth torsion [1, Theorem 5,1]. Homotopy invariance can be verified using the fact that the construction of $\varrho(p)$ involves only chain complexes associated to p . Using Lemma 10.1 we obtain that $[\varrho(p)] = 0$ if p is either a product bundle or a rational homotopy equivalence. By Proposition 9.5 and (9.6) we get then that $[\varrho(p)] = 0$ for all unipotent bundles p . \square

Proof of Theorem 1.1. Combining Theorems 1.4 and 1.3 we obtain the statement of Corollary 1.5: for any unipotent bundles $p: E \rightarrow B$ and $q: D \rightarrow E$ there is a homotopy

$$\tau^s(pq) \simeq \mathrm{Wh}_s^{\mathbb{Q}}(q^!)\tau^s(p)$$

In [1, Theorem 7.1] an analogous decomposition of the smooth torsion of pq (in the case where q is a Leray-Hirsch bundle) was the main ingredient in the proof of the fact that the cohomological torsion of p and q satisfies the formula (1-1). The same argument can be now used to obtain the formula (1-1) for arbitrary unipotent bundles p and q . \square

11. APPENDIX: MAPS OF HOMOTOPY FIBERS

Multiple arguments in this paper involve constructions of maps between homotopy fibers as well as constructions of homotopies between such maps. We summarize here the basic scheme of such constructions.

11.1. Maps of homotopy fibers. For a space X with a basepoint x_0 let $P_{x_0}X$ denote the space of paths in X that start at x_0 . By the homotopy fiber of a map $p: Y \rightarrow X$ over x_0 we understand the standard construction

$$\mathrm{hofib}(p)_{x_0} := \{(\omega, y) \in P_{x_0}X \times Y \mid \omega(1) = p(y)\}$$

We will denote by $i_Y: \mathrm{hofib}(p)_{x_0} \rightarrow Y$ the map given by $i_Y(\omega, y) = y$.

Assume that we have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{f}} & Y' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{f} & X' \end{array}$$

such that $f(x_0) = x'_0$. Given a homotopy h from fp to $p'\tilde{f}$ we obtain a map $\tilde{\tilde{f}}: \mathrm{hofib}(p)_{x_0} \rightarrow \mathrm{hofib}(p')_{x'_0}$ given by

$$\tilde{\tilde{f}}(\omega, y) := (f\omega * h_y, \tilde{f}(y))$$

Here h_y denotes the path in X' defined by $h_y(t) = h(y, t)$, and $*$ indicates concatenation of paths. We have

$$i_{Y'} \tilde{\tilde{f}} = \tilde{f} i_Y$$

11.2. Homotopies of maps of homotopy fibers. Assume that that we have two diagrams:

$$(11-1) \quad \begin{array}{ccc} Y & \xrightarrow{\tilde{f}_0} & Y' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{f_0} & X' \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{\tilde{f}_1} & Y' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{f_1} & X' \end{array}$$

that commute up to homotopies h_0 and h_1 , respectively. As a result we get two maps of the homotopy fibers:

$$\tilde{\tilde{f}}_0, \tilde{\tilde{f}}_1: \mathrm{hofib}(p)_{x_0} \rightarrow \mathrm{hofib}(p')_{x'_0}$$

In order to obtain a homotopy between these maps it suffices to construct the following data:

- 1) a homotopy $\tilde{H}: Y \times I \rightarrow Y'$ between \tilde{f}_0 and \tilde{f}_1 ;
- 2) a basepoint preserving homotopy $H: X \times I \rightarrow X'$ between f_0 and f_1 ;

- 3) a homotopy of homotopies between the two homotopies $p' \tilde{f}_0 \simeq f_1 p$: the one given by concatenating $p' \tilde{H}$ with h_1 and the one obtained by concatenating h_0 with $H(p \times \text{id}_I)$:

$$\begin{array}{ccc} p' \tilde{f}_0 & \xrightarrow{p' \tilde{H}} & p' \tilde{f}_1 \\ h_0 \downarrow & & \downarrow h_1 \\ f_0 p & \xrightarrow{H(p \times \text{id}_I)} & f_1 p \end{array}$$

Notice that giving such a homotopy of homotopies is equivalent to giving a map

$$\Theta: Y \times I \times I \rightarrow X'$$

such that $\Theta|_{Y \times \{i\} \times I} = h_i$ for $i = 0, 1$, $\Theta_{Y \times I \times \{0\}} = H(p \times \text{id}_I)$, and $\Theta_{Y \times I \times \{1\}} = p' \tilde{H}$. The homotopy $\tilde{\tilde{H}}$ between \tilde{f}_1 and \tilde{f}_2 is defined by:

$$\tilde{\tilde{H}}((\omega, y), t) := (H_t \omega * \Theta_{y,t}, \tilde{H}(y, t))$$

where $H_t: X \rightarrow X'$ is given by $H_t(x) = H(t, x)$ and $\Theta_{y,t}$ is the path in X' given by $\Theta_{y,t}(s) = \Theta(y, t, s)$.

11.3. An obstruction to lifting a homotopy. Assume again that we have two homotopy commutative squares (11-1) and that \tilde{f}_0, \tilde{f}_1 are the maps of homotopy fibers defined by these squares. Assume also that we have a homotopy \tilde{H} between \tilde{f}_0 and \tilde{f}_1 , and that we want to determine whether there exists a homotopy $\tilde{\tilde{H}}$ between \tilde{f}_1 and \tilde{f}_2 that fits into a commutative diagram

$$(11-2) \quad \begin{array}{ccc} \text{hofib}(p)_{x_0} \times I & \xrightarrow{\tilde{\tilde{H}}} & \text{hofib}(p')_{x'_0} \\ i_Y \times \text{id}_I \downarrow & & \downarrow i_{Y'} \\ Y \times I & \xrightarrow{\tilde{H}} & Y' \end{array}$$

In (11.2) we described data that suffices to construct such $\tilde{\tilde{H}}$, but here we are interested in a condition that is equivalent to the existence of this homotopy. We can describe such a condition as follows.⁶

Let $C_p: \text{hofib}(p)_{x_0} \times I \rightarrow X$ be the map given by $C_p((\omega, y), t) = \omega(t)$. This is a homotopy between the constant map into x_0 and the map pi_Y . Consider the

⁶ See also [15, Lemma 4.1]

following diagram:

$$\begin{array}{ccc}
 p' \tilde{f}_0 i_Y & \xrightarrow{p' \tilde{H}(i_Y \times \text{id}_I)} & p' \tilde{f}_1 i_Y \\
 h(i_Y \times \text{id}_I) \downarrow & & \downarrow h'(i_{Y'} \times \text{id}_I) \\
 f_0 p i_Y & & f_1 p i_{Y'} \\
 f_0 C_p \searrow & x'_0 & f_1 C_p \swarrow
 \end{array}$$

Each vertex of this diagram represent a map $\text{hofib}(p)_{x_0} \rightarrow X'$ and edges represent homotopies of such maps. Concatenating all homotopies appearing here we obtain a homotopy from the constant map into x'_0 to itself, or equivalently a map

$$\varphi: \text{hofib}(p)_{x_0} \rightarrow \Omega X'$$

It is straightforward to verify that the map φ is contractible iff there exists a homotopy \tilde{H} such that the diagram (11-2) commutes. In other words the homotopy class of φ is an obstruction to lifting the homotopy \tilde{H} to a homotopy defined on the level of the homotopy fibers.

Note. Let $p': (Y', y'_0) \rightarrow (X', x'_0)$ be a map of infinite loop spaces where x'_0, y'_0 are the trivial elements in X' and Y' . In this case the map φ has a simpler interpretation. Namely, let $j_{Y'}: \Omega X' \rightarrow \text{hofib}(p')_{x'_0}$ be the map given by $j_{Y'}(\omega) = (\omega, y_0)$. The set of homotopy classes $[\text{hofib}(p)_{x_0}, \text{hofib}(p')_{x'_0}]$ has a structure of an abelian group and we have:

$$j_{Y'*}[\varphi] = [\tilde{f}_1] - [\tilde{f}_2]$$

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